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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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I.

A. THE PURPOSE OF ZENO'S ARGUMENTS.

Introduction. No questions on the foundations of mathematics are as old and of such perennial interest, reaching into the most recent speculations on the philosophy of mathematics, as are Zeno's arguments on motion. Zeno flourished in the fifth century before Christ, but only recently has G. Cantor's *Mengenlehre* been applied to the fuller elucidation of Zeno's paradoxes. The history of these paradoxes is largely the history of concepts of continuity, of the infinite and infinitesimal.

There has been great difference of opinion as to the exact nature and purpose of Zeno's arguments. None of Zeno's writings have come down to us. We know of his tenets only through his critics and commentators—Plato, Aristotle and Simplicius. Plato was born about 60 years after Zeno, Aristotle about 100 years after. Simplicius lived nearly 1,000 years after Zeno. Plato does not reproduce Zeno's arguments,[1] but discusses their purpose, which was "to protect the arguments of Parmenides against those who make fun of him and seek to show the many ridiculous and contradictory results which they suppose to follow from the affirmation of the *one*"; Zeno argues that "there is no *many*," he "denies plurality."

Aristotle's Interpretation of Zeno's Arguments. Aristotle gives in very compressed form the arguments on motion, as they were handed down to him, in these words:[2]

[1] Plato, *Parmenides*, 127 D., Jowett's translation.

[2] Aristotle, *Physics*, VI, 9. See Carl Prantl's edition, in Greek and German, Leipzig, 1854. We omit the latter part of Prantl's version of Zeno's fourth argument against motion. The text is defective. Later we give Burnet's elaboration of the proof, which embodies the most probable conjecture of what Aristotle originally said.

"Zeno reasons here incorrectly; for, he says that everything, when in a uniform state, is continually either at rest or in motion, and that a body moving in space is continually in the *Now* [the instant], hence the arrow in its flight is at rest. But this is false, for the reason that time is not composed of individual, indivisible *Nows*, as also no other quantity is so composed. There are four proofs advanced by Zeno against motion, which present many difficulties to those who try to refute them. The first is the one on the impossibility of motion, on the ground that a thing moving in space must arrive at the mid-point before it reaches the end-point. We have gone into the details of this matter in our previous discussion. The second is the so-called Achilles; it consists in this that in a race the faster cannot overtake the slower; for, the pursuer must always first arrive at the point from which the one pursued has just departed, so that the slower is necessarily always a small distance ahead. But this is the same argument as that of bisection and differs from that merely in this, that the distance added is not divided quite into halves. That the slower is not overtaken follows from this argument, but it rests upon the same assumption as the bisection (for in both arguments it is stated that a thing cannot reach the end-point, since the quantity is divided in some manner. However, this second argument has the additional contention, that, in a race, even the most rapid cannot overtake the slowest and the refutation must therefore be the same. The claim that the one in the lead cannot be overtaken is false. To be sure, in the moment when he has the lead, he is not overtaken. Nevertheless he is overtaken; Zeno merely admits that the pursuer completely passes over the entire distance. These are two of his proofs; the third is the one referred to above, that the moving arrow is at rest. It is based on the assumption that time is made up of the individual *Nows*. If this is not admitted, then the conclusion does not follow. The fourth is in regard to equal bodies which move on a track parallel to other bodies of equal size but moving in opposite directions, namely the first moving thither from the end of the track, the second moving hither from the middle of it with the same speed. From this he thought that he must conclude that the half time must be equal to its double. The fallacy lies in the claim that when a body moves parallel to one in motion, with the same speed as it does move, passes one that is at rest, the time of passing is the same in both cases. This is false."

For greater clearness we repeat Zeno's arguments in the expanded form given by Burnet,[1] which is a free paraphrase of Aristotle's statements. We shall find it convenient, for future reference, to use the names "Dichotomy," "Achilles," "Arrow," and "Stade" for the four arguments against motion, respectively.

1. "DICHOTOMY": You cannot traverse an infinite number of points in a finite time. You must traverse the half of any given distance before you traverse the whole, and the half of that again before you can traverse the whole, and the half of that again before you can traverse it. This goes on *ad infinitum*, so that (*if space is made up of points*) there are an infinite number in any given space, and it cannot be traversed in a finite time.
2. "ACHILLES": The second argument is the famous puzzle of Achilles and the tortoise. Achilles must first reach the place from which the tortoise started. By that time the tortoise will have got on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, but he never makes up to it.
3. "ARROW": The third argument against the possibility of motion *through a space made up of points* is that, on this hypothesis, an arrow in any given moment of its flight must be at rest in some particular point.[2]

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[1] J. Burnet, *Early Greek Philosophy*, 1892, pp. 322 ff.

[2] A different version of the "arrow" is given by Diogenes Laertius, IX, 72, who lived 500 years after Aristotle, probably about 200 A.D.: "That which moves can neither move in the place where it is, nor yet in the place where it is not." In expanded form this argument is given by William Minto, professor of logic in the University of Aberdeen, in his *Logic, Inductive and Deductive*, London, 1893, p. 224, and by W. R. Royce Gibson in his *The Problem of Logic*, London, 1908, p. 290, as follows:

"If a body moves, it must move either where it is or where it is not." "But a body cannot move where it is; neither can it move where it is not."

"Therefore, it cannot move at all; i. e. motion is impossible."

4. "STADE": Suppose three parallel rows of points in juxtaposition, as in Fig. 1.

FIG. 1.	FIG. 2.
A • • • • • • •	← A • • • • • • •
B • • • • • • •	B • • • • • • •
C • • • • • • •	C • • • • • • • • →

One of these (B) is immovable, while A and C move in opposite directions with equal velocity so as to come into the position represented in Fig. 2. The movement of C relatively to A will be double its movement relatively to B, or, in other words, any given point in C has passed twice as many points in A as it has in B. It cannot, therefore, be the case that an instant of time corresponds to the passage from one point to another.

Tannery's Interpretation of Zeno's Arguments. As reported by Aristotle and Simplicius, Zeno's arguments are fallacies. That Zeno's reasoning was wrong has been the view universally held since the time of Aristotle down to the middle of the nineteenth century. During these many centuries the efforts of philosophers and mathematicians on this matter were to explain the exact nature of Zeno's blunders. More recently the opinion has been advanced that Zeno was incompletely and incorrectly reported, that his arguments were turned away from their true purpose by the Sophists who used them in advancing skepticism and the denial of knowledge, and that Aristotle described them as modified by the Sophists. Three great leaders in the interpretation of Greek thought, Cousin, Grote and P. Tannery, have construed Zeno's arguments as serious efforts, conducted with logical rigor. Cousin[1] maintained that Zeno successfully opposed the idea of *multiplicity devoid of all unity*. Grote[2] held a similar view. Tannery[3] argued that Zeno opposed the idea that *a point is unity in position*. Zeller[4] rejects all three explanations, mainly on the ground that they do not find support in the extant writings of Greek philosophers. It is also true that no Greek account definitely refutes any of the three interpretations. This lack of exact and detailed information is deplorable. Tannery's resuscitation of Zeno's arguments deserves our attention because of the internal coherence imparted to them. According to Tannery, Zeno did not deny motion, but wanted to show that motion was impossible under the conception of space as the sum of points. Tannery does not battle against the traditional statement that Zeno argued against plurality; he accepts Plato's general explanation, but differs from him and others on the precise nature of this plurality. According to Tannery it was not the ordinary notion that Zeno combated, according to which two lambs

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This version of the "Arrow" must be rejected for two reasons: First, it occurs for the first time about 700 years after Zeno and is for that reason unreliable; second, there is no kernel to the argument. As Gibson says, it amounts to this: "If a body moves, it must move under conditions which render motion impossible."

[1] *Fragments philosophiques*, par M. Cousin. 5ème éd., Paris, 1865, p. 69.

[2] George Grote, *Plato*, Vol. I, 3d ed., London, 1875, pp. 100–104.

[3] Paul Tannery, "Le concept scientifique du continu. Zénon d'Elée et Georg Cantor," *Revue philosophique de la France et de L'Etranger*, X année, T. XX (1885), pp. 385–410; Paul Tannery, *Science hellène*, Paris, 1887, pp. 247–261.

[4] E. Zeller, *Die Philosophie der Griechen*, 1. Theil, 1. Hälfte, 5. Aufl., Leipzig, 1892, pp. 591–604.

are not one lamb, but a special notion of the Pythagoreans. Zeno's master, Parmenides, was attacked (says Tannery) by the Pythagoreans, and Zeno stepped into the conflict by battling against the mystical Pythagorean idea of a mathematical point—a point defined as *unity having position.* This Pythagorean definition is mentioned by Aristotle. Tannery interpreted this definition as signifying that a solid is the sum of points, just as a number is the sum of units. But such an idea is false. A point, mathematically speaking, is not unity or 1; it is a pure zero or 0. This interpretation of the phrase *unity having position* attributes to Zeno the grasp of an abstract concept, namely that of a point, destitute of length, breadth and thickness. That it is not unreasonable to ascribe to Zeno such abstractions seems evident from the following passage in Aristotle:[1]

> "If the absolute unit is indivisible it would be, according to Zeno's axiom, nothing at all. For that which neither makes anything larger by its addition, nor makes anything smaller by its subtraction, is not one of the things that are, since it is clear that what *is* must be a magnitude, and, if a magnitude, corporeal, for the corporeal has being in all dimensions. Other things, such as the surface and the line, when added in one way make things larger, when added in another way do not; but the point and the unit do not make things larger however added."

It is hard to make out how much of this is the thought of Aristotle, and how much of it is Zeno's. Yet there would be no occasion to mention Zeno, had he no share in it.

It is believed by many critics that Zeno gave his arguments in the form of dialogue. Acting upon this view, Tannery entered upon the reconstruction of Zeno's arguments from the compressed passages handed down to us. Consider the following argument of Zeno on divisibility, as stated by Simplicius:[2]

> "If that which is, has no magnitude it could not even be. Everything that truly is must needs have magnitude and thickness, and one part of it must be separated from another by a certain interval. And the same may be said of the next smaller part; it too will have magnitude, and a next smaller part. As well say this once for all as keep repeating it forever. For there will be no such part that could serve as a limit. And there will never be one part save in reference to another part. Thus, if the many have being, they must be both large and small—so small as to have no size at all, and so large as to be infinite."

Tannery's reconstruction of this passage is as follows:

A Pythagorean adversary claims that a finite quantity can be regarded as the sum of indivisible parts.

Zeno presents the first part of the dilemma resulting therefrom, thus: Admitting, as both of us do, that a quantity is infinitely divisible by continued bisection, it is evident that the parts become smaller and smaller. Hence, if there is a last term, it is 0. But the sum of such indivisible terms 0 is only 0. Hence the quantity has no magnitude.

But, says his adversary, why may the indivisible parts not be different from 0 and have magnitude?

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[1] Aristotle, *Met.*, II, 4, 1001 b 7; translation taken from C. M. Bakewell, *Source Book in Ancient Philosophy*, New York, 1907, p. 23.

[2] Simpl. 140, 34 [R. P., 1050, Fr. 2 in Diels' arrangement]; C. M. Bakewell, *op. cit.*, p. 22.

Then Zeno presents the second part of the dilemma: If the indivisible parts have magnitude, and are infinite in number, the sum of these parts must be infinite.

Consequently, a finite quantity cannot be regarded as the sum of indivisible parts.

This explanation of Zeno's argument places Zeno certainly higher as a logician than does the old explanation which charged Zeno with inability to see that, if $xy = c$, x can increase and y simultaneously decrease in such a way that their product remains the same.

Now, let us see how Tannery applies *a point as unity in position* to the resuscitation of Zeno's arguments on motion. Tannery presents them in the form of a double dilemma.

The first argument, the "Dichotomy," involves matters which we have considered above, in connection with infinite division. As long as space is assumed to be made up of indivisible parts, the infinite number of parts, admitted by both contestants to result from continued bisection, cannot all be passed over in a given time.

The adversary may now present the point advanced by Aristotle, that the bisection is not carried on to actual infinity, but only to a potential infinity, and may therefore be run over in a finite time.

Zeno replies by stating the "Achilles," which does not involve bisection and in which the time-interval is subdivided in much the same way as the space-interval.

The adversary then takes the position that he has admitted too much. Finite time, he claims, is capable of division into an infinity of parts. Is there not a sum of instants? May there not correspond an instant to each successive position?

Against this Zeno directs his last two arguments, which constitute a second dilemma. At each instant the flying arrow occupies a fixed position. But occupying a fixed position at a given instant means that it is at rest that instant. Hence the arrow is at rest every instant of its flight.

The adversary explains that when saying that time was the sum of instants, he did not mean that each instant should apply to a fixed position of the arrow, but rather to the passage from each position to the next following position.

Here Zeno advances his "Stade" as his fourth argument. He shows that the demand of his adversary cannot be granted, because it would make all motions equal.

A motion from a point A (see Fig. 1 and Fig. 2) to the next point on the left requires one instant.

A motion from a point C to the next point on the right requires the same instant.

Hence A moves relatively to C twice as fast as relatively to B.

It is therefore not the passage from one point to the next that corresponds to the instant, for it would then follow that one is equal to its double.

Tannery's explanation of the four arguments, particularly of the "Arrow" and "Stade," raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force. It does not place Zeno in the position of being ignorant of the most simple ideas of relative motion; it exhibits Zeno as a logician of the first rank.

Tannery's conclusions have been strongly supported by G. Milhaud,[1] but opposed by other French writers and by Zeller.

CENTERS OF SIMILITUDE OF CIRCLES AND CERTAIN THEOREMS ATTRIBUTED TO MONGE. WERE THEY KNOWN TO THE GREEKS?

By RAYMOND CLARE ARCHIBALD, Brown University.

One of the most noticeable characteristics of French, German and Italian, as opposed to American, texts on elementary geometry is the emphasis laid on broad underlying principles. How many American high-school graduates could give one any idea of the theory of similitude of plane and solid figures? How many teachers realize the importance of this far reaching theory in the solution of geometrical problems, or are familiar with the equivalent of Petersen's excellent exposition?[2] At all events it seems well worth while to draw attention to some simple results in the theory, and to put on record their historical setting. The theorems attributed to Monge, which I propose to discuss, involve the centers of similitude of circles (spheres).

The centers of similitude of two circles (spheres), whose centers are A and B, are the points which divide AB internally, at C, and externally, at D, in the ratio of the radii, r_a, r_b ($r_a \geqq r_b$);

$$AC : CB = AD : BD = r_a : r_b.$$

The circles (spheres) may be situated in any fashion. If they are tangent (internally or externally), the point of tangency is a center of similitude. If concentric, we may, perhaps, say that either A, B, C, D coincide or else A, B, C coincide while D is indeterminate. If they are equal and non-concentric, D is at infinity[3] and C bisects AB.

Lines joining the ends of parallel radii pass through a center of similitude, and common tangent lines (planes), when they exist, also pass through such a center. Conversely, if through a center of similitude, D (or C), of two circles a line be

[1] G. Milhaud, "Le concept du nombre chez les Pythagoriciens et les Éléates," *Revue de métaphysique et de morale*, I, Paris, 1893, p. 141.

[2] J. Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879, pp. 22 ff. This is an English edition of the remarkable Danish original. There are also French, German, Italian, Hungarian and Russian translations making in all some 15 editions. Because of its many notable qualities, this work stands preeminent in its special field.

[3] This is, of course, more an idea of *projective*, than of *elementary*, geometry.

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II.

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B. ARISTOTLE'S EXPOSITION AND CRITICISM.

The purposes of Zeno's arguments, as set forth by Cousin, Grote, and P. Tannery differed from the purpose as it was understood by Plato, Aristotle, and the later Greek writers. The three modern interpreters gave to the arguments settings in the history of Greek thought which exhibit them as flawless in logical rigor. On the other hand, Aristotle and subsequent Greek writers interpreted the arguments as fallacies and expended their mental acumen in attempts to point out the real nature of the fallacies. Except for the recent studies of Cousin, Grote, and Tannery, the history of Zeno's arguments against motion has been for over 2,000 years the history of attempts to explain Zeno's "fallacies." Aristotle acknowledged the great difficulty in exposing their hidden source of logical error. The sixth book of his *Physics* is devoted to the exposition of the subtle notions of continuity and infinity. In fact, the entire *Physics*, of the size of about 225 ordinary modern pages of print, gives all of its eight books to discussions of the notions of motion, divisibility, continuity, infinity, and the vacuum. As one reads the *Physics* one is impressed by the fact that the great Stagirite did not hesitate to restate his arguments repeatedly. For twenty centuries Aristotle's criticisms of Zeno have been the starting points of philosophic discussion; for twenty centuries Zeno's fallacies have puzzled many of the best minds. Like the problems of the trisection of an angle, of the squaring of the circle, and of the duplication of the cube, Zeno's fallacies have challenged some of the best brains; but, like the former problems, have finally been forced to surrender their secrets and to enter the group of "problems of the past."

Before entering upon the study of Aristotle we must refer to the atomistic

theory advanced by Leucippus (about 500 B. C.) and developed further by Democritus (about 460 to about 357 B. C.) and others. They conceived magnitudes as composed of indivisible elements in finite number. This view must have gained some ascendancy among a few early mathematicians. Democritus himself was not without mathematical ability. From a notice of Plutarch, Democritus raised the following question:[1]

"If a cone were cut by a plane parallel to its base,[2] what must we think of the surfaces of the sections, that they are equal or unequal? For if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal the sections will be equal, and the cone will appear to have the property of a cylinder, viz., to be composed of equal, and not unequal, circles, which is very absurd."

This paradox of Democritus places difficulties in the way of accepting the notion of an infinitesimal and thereby indirectly favors the idea of divisibility in only a finite number of parts.

In Antiphon's attempt to square the circle it is assumed that straight and curved lines are ultimately reducible to the same indivisible elements. Antiphon, according to the testimony of Simplicius and Philoponus, inscribed in a circle a square and, by arc-bisection, obtained regular polygons of 8, 16, 32 sides, and so on. He assumed that a polygon could be reached that coincides with the circle. Simplicius observes:[3]

"The conclusion here is manifestly contrary to geometrical principles, not, as Alexander maintains, because the geometer supposes as a principle that a circle can touch a straight line in one point only, and Antiphon sets this aside; for the geometer does not suppose this, but proves it. It would be better to say that it is a principle that a straight line cannot coincide with a circumference, for one without meets the circle in one point only, one within in two points, and not more, and the meeting takes place in single points. Yet, by continually bisecting the space between the chord and the arc, it will never be exhausted, nor shall we ever reach the circumference of the circle, even though the cutting should be continued *ad infinitum:* if we did, a geometrical principle would be set aside, which lays down that magnitudes are divisible *ad infinitum*."[4]

Aristotle's idea of continuity differs from the idea of continuity as developed by Georg Cantor and his followers. Aristotle's is a sensuous, physical continuum, in which there is an intimate bond between its elements; Cantor's is a collection of elements, arranged in order, infinite in number, but externally to each other; it is purely abstract and transcends the power of the imagination to grasp it. As Aristotle's *Physics* is not available in English translation, we shall translate from it rather freely. Says Aristotle:[5]

"If things continuous, things touching one another, and things successive are as described above, namely that continuous things are those whose extreme limits are one [united], things

[1] Plutarch, *de Comm. Not.*, Vol. IV, p. 1321, ed. Didot. Quoted from Allman, *Greek Geometry*, 1889, p. 81.

[2] This passage obviously means, that the cutting plane is infinitely near to the base of the cone.

[3] *Simplicii comment. in octo Aristotelis physicae auscultationis libros.* Venetiis, 1526, § 80. We quote the translation given in G. J. Allman's *Greek Geometry*, 1889, p. 66.

[4] According to another commentator, Themistius (317–387 A. D.), Antiphon began his process of squaring the circle, not by inscribing a square, but by inscribing an equilateral triangle.

[5] Aristotle's *Acht Bücher der Physik.* Griechisch und Deutsch von Carl Prantl, Leipzig, 1854, p. 279, Buch VI, § 1.

touching are those whose extreme limits have the same position, things successive are those between which there lies nothing that is like them, then it is impossible that a thing continuous consist of indivisible things, as for instance, that a line be made up of points, the line being taken as continuous, and a point as indivisible."

This passage is made clearer by a quotation or two from the *Physics*, Bk. V, § 3:

"Continuous are those things which in their union become one in nature; and just as that which holds them in continuity together becomes one, so the whole will be one, as for instance by means of a nail, or glue, or an adherence or an accretion."

"Things continuous must necessarily touch; on the other hand, things touching are not necessarily continuous, for the extreme ends, even though locally together, are not necessarily one and the same."

Returning to the *Physics*, Bk. VI, § 1, we reproduce the chief part of Aristotle's argument which leads him to the conclusion that a line is not made up of points:

"For neither are the extreme ends of points one [united], for the reason that, of an indivisible, the one cannot be an extreme and the other another part, nor are the extreme ends locally together, since the indivisible can have no extreme."

According to Aristotle, things continuous are always divisible into parts that are continuous. In the same way, time is not made up of parts considered as indivisible *Nows*.[1] We quote from *Physics*, Bk. VI, § 2:

"As each magnitude is divisible into magnitudes (for it has just been proved that the continuous cannot consist of the indivisible, and every magnitude is continuous), the faster of two must necessarily in the same time be moved through a larger distance and in a less time through an equal distance, and also in a less time through a larger distance, as indeed some define the faster. . . ."

"Since all motion takes place in time, and in each time it is possible for something to be moved and everything movable can be moved faster and also slower, then there can occur in each time the faster and the slower motion. If this is so, then time must be continuous; by the continuous I mean that which is always divisible into divisible parts. . . ."

"If time is continuous, so is distance, for in half the time a thing passes over half the distance, and, in general, in the smaller time the smaller distance, for time and distance have the same divisions; and if one of the two is unlimited, so is the other. . . . For that reason the argument of Zeno assumes an untruth, that one unlimited cannot travel over another unlimited along its own parts, or touch such an unlimited, in a finite time; for length as well as time and, in general, everything continuous, may be considered unlimited in a double sense, namely, according to the [number of] divisions or according to the [distances between the] outermost ends."

This profound criticism directed against Zeno refers evidently to the "Achilles" where we are in danger of modifying, in our minds, the conditions actually existing; we are in danger of thinking of the distance between Achilles and the tortoise as decreasing through an unlimited number of subdivisions, while the time for traversing these successive subdivisions is thought of as about the same for each. There results from this distortion of events an unlimited subdivision of a finite distance, and an unlimited accumulation of finite time-intervals; the former yields what from a modern view-point constitutes a convergent geometric series of space-intervals; the latter a divergent series of time-intervals. A finite distance is made to be traversed in an infinite time.

Aristotle's arguments against the "Achilles" and the "Dichotomy" are the

[1] See *Physics*, Bk. VI, § 9, quoted earlier in this article.

same. He touches upon them in other parts of his *Physics*. Since a line cannot be built up from points, a line cannot actually be subdivided into points.[1]

"The continued bisection of a quantity is unlimited, so that the unlimited exists potentially, but is actually never reached."[2]

"Previously we refuted this [the "Dichotomy"] by the fact that time has unlimitedly many parts, in consequence of which there is no absurdity in the consideration that in unlimited time-intervals one passes over unlimitedly many spaces . . . [but as Aristotle does not consider this a full explanation, he continues:] If one divides a continuous line into halves, he uses one point at two, for he makes it the beginning [of one part] and the end [of the other]; in this manner proceeds he who . . . bisects, but in this division neither the line nor the motion are continuous; for . . . in the continuous there are, to be sure, unlimitedly many halves, but not actually unlimited, only potentially so."[3]

Of the numerous passages in Aristotle's *Physics* which might be quoted as bearing on the "Arrow" we choose the following (Bk. VI, § 8):

". . . A thing is at rest, when it is unchanged in one *Now* and still in another *Now*, it itself as well as its parts remaining in the same status. . . . There is no motion nor rest in the *Now* . . . In a time-interval, on the contrary, it [a variable] cannot exist in the same state of rest, for otherwise it would follow that the thing in motion is at rest."

Aristotle's disproof of the "Stade" is given at the end of our first extract from the *Physics* (Bk. VI, § 9), where it is pointed out that Zeno falsely assumes a body to move with the same velocity relative to a body that is at rest, as it would with respect to one in motion.

Further Comments on Aristotle. The main criticism to be made on the profound arguments of Aristotle is that they do not go far enough. The "Dichotomy" and "Achilles" involve the theory of limits, a theory which in very recent time has been found to be in need of reconstruction and which was imperfectly developed in Greek antiquity. Particularly insistent in our mind is the query raised by Zeno's arguments, *how is it possible for a variable to reach its limit?* This query finds no reply in Aristotle. To be noted is also the fact that Aristotle denied the existence of actual infinity, as distinguished from potential infinity. The "Arrow" called for a sharp definition of the *Now* (the instant). Has the instant no duration? Is it, so to speak, a point of time? Or is the instant an infinitesimal—some constant different from zero, yet smaller than any finite quantity? Here Aristotle drew a sharp line; the *Now* was a point of time; it had no duration; in the *Now* there could be neither motion nor rest.

It will be seen that the mathematical concepts herein involved are most fundamental. The concept of a limit involves notions of infinity and is historically connected with the concept of the infinitesimal. All these concepts are basic. It would be difficult to select three other notions more far-reaching in mathematical science than *infinitesimal, infinity, limit.*

[1] Aristotle, *Physik*, VI, 1, Prantl's ed., p. 281.

[2] Aristotle, *Physik*, III, 7, Prantl's ed., p. 141.

[3] Aristotle, *Physik*, VIII, 8, Prantl's ed., pp. 445, 447. See also Aristotle's *Lib de lineis insecab.*, p. 968.

III.

C. A Two-Thousand Year Struggle for Light.

1. The Greeks after Aristotle.

Whether Plato and Aristotle correctly explained the nature of Zeno's arguments and the purpose which Zeno himself had in mind in presenting them, is of no concern in tracing the history of thought on this subject after the time of Aristotle. Apparently the writings of Plato and Aristotle constituted the sources of information for later writers. Zeno's arguments were given by Aristotle in the form of "fallacies," and it is the influence of these "fallacies" that remains to be traced. There is little doubt that this influence was great upon the development of Greek geometry. Since Aristotle, with all his dialectical skill, was not able to satisfactorily explain all the paradoxes which had arisen in the study of the infinite and of motion, the conclusion of H. Hankel[1] and other recent historians of mathematics is probably correct, that the infinite and infinitesimal were banished from the classic Greek geometry for the sake of greater rigor. We shall see that recently discovered manuscripts of Archimedes confirm this view. Mathematicians assumed that every magnitude is divisible at pleasure. The doctrine of incommensurable lines rests upon the possibility of unlimited divisibility. The denial of the existence of the infinitesimal goes back to Zeno who is reported by Simplicius[2] to have stated: "That which, being added to another, does not make it greater, and being taken away from another does not make it less, is nothing." This momentous question presented itself twenty-two centuries later to Leibniz who gave different answers. In one exposition Leibniz extended the definition of equality so as to declare magnitudes as equal when they differ from one another by an incomparably small quantity. The later Greek mathematicians followed a radical policy toward the infinitesimal; they formally excluded it from demonstrative geometry by a postulate. This was done by Eudoxus (408–355 B. C.), by Euclid (about 300 B. C.) and by Archimedes (287–212 B. C.). Archimedes gives the postulate, which he attributes to Eudoxus, as follows:[3]

"When two spaces are unequal, it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded."

Euclid in his *Elements* (Bk. V, Def. 4) gives the postulate in the form of a definition:

"Magnitudes are said to have a ratio to one another, when the less can be multiplied so as to exceed the other."

The Method of Archimedes, a book formerly thought to be irretrievably lost, but fortunately discovered by Heiberg in 1906 in Constantinople, gives inter-

[1] H. Hankel, *Geschichte der Mathematik im Alterthum und Mittelalter*, Leipzig, 1874, p. 120.

[2] Simplicius, *Phys.* 30*a*. Quoted by E. Zeller, *History of Greek Philosophy*, Vol. 1, London, 1881, p. 615.

[3] Archimedes, *De quadr. parabol. Praef.*

esting evidence that the notion of infinitesimals, though not used by Archimedes in formal demonstrations, was employed by him in tentative research as a method of discovery. He considered infinitesimals sufficiently scientific to suggest the truths of theorems, but not to furnish rigorous proofs. The process is mechanical, consisting of the weighing of infinitesimal elements, which he calls straight lines or plane areas, but which are really infinitely narrow strips or infinitely thin plane laminæ.[1] The breadth or thickness is regarded as being the same in the elements weighed at any one time.

Further evidence that infinitesimals lingered in the minds of Greek thinkers through the centuries is furnished by the skeptic, Sextus Empiricus (200 A. D.), who advances the paradox that, when a line rotating in a plane about one of its ends describes a circle with each of its points, these concentric circles are of unequal area, yet each circle must be equal to the neighboring circle which it touches.[2] The difficulty encountered here is similar to that raised by Democritus over 500 years earlier.

An interesting remark about Zeno is made by Plutarch (about 90 A. D.) in his life of Pericles. He says that Pericles was also a hearer of Zeno, the Eleatic, who "also perfected himself in an art of his own for refuting and silencing opponents in argument; as Timon of Phlius describes it—

> Also the two-edged tongue of mighty Zeno, who,
> Say what one would, could argue it untrue."

Thus we see that the personality of Zeno and some of the ideas involved in his "paradoxes" appear here and there on the surface of Greek thought, thereby indicating an underground flow of those ideas down the centuries of Greek history.

Sextus Empiricus also gives a version of the "Arrow" much like what we quoted from Diogenes Laertius. He attributes the paradox not to Zeno but to Diodorus Cronus. The existence of motion is disproved thus: If matter moves, it is either in the place in which it is, or the place in which it is not; but it cannot move in the place in which it is, and certainly not in the place in which it is not: hence it cannot move at all. To this Sextus Empiricus replies by stating another argument equally paradoxical and therefore far from illuminating: By the same rule men never die, for if a man die, it must either be at a time when he is alive, or at a time when he is not alive; hence he never dies.

2. The Romans.

The subtleties of Zeno's arguments on motion attracted little attention among the Romans. Lucretius (96–55 B. C.) used the notion of infinity in arguments on atomic theory. He reasoned that one must assume the existence of atoms (indivisible, but not mathematical points), otherwise each body,

[1] T. L. Heath, *Method of Archimedes*, Cambridge, 1912, p. 8.

[2] Sextus Empiricus, *Adv. math.*, I, III, § 66 ff., ed. Fabricius. p. 322; referred to in K. Lasswitz, *Geschichte der Atomistik* I, Hamburg u. Leipzig, 1890, p. 148.

whether large or small, would consist of an infinite number of parts, and there would be no difference between the largest and the smallest, both being infinity.[1]

Cicero and Seneca mention Zeno in passing: the one to display Zeno's love for argument even though faulty, the other to display Zeno's skepticism. Cicero[2] attributes to Zeno the following pointed syllogism: "That which exercises reason is more excellent than that which does not exercise reason; there is nothing more excellent than the universe; therefore the universe exercises reason." Seneca[3] exhibits Zeno as denying not only plurality as did his master, but denying also unity and the real existence of external objects. We have no good reason for accepting either Cicero's or Seneca's view of Zeno's aim and purpose.

3. Medieval Times.

The earliest church father known to interest himself in arguments on motion was St. Augustine (354–430 A. D.). In a dialogue on the question, whether or not the mind of man moves when the body moves, and travels with the body, he is led to a definition of motion, in which he displays some levity. It has been said of scholasticism that it had no sense of humor. Hardly does this apply to St. Augustine in his discussion of the impossibility of motion. He says:

> "When this discourse was concluded, a boy came running from the house to call us to dinner. I then remarked that this boy compels us not only to define motion, but to see it before our very eyes. So let us go, and pass from this place to another; for that is, if I am not mistaken, nothing else than motion."[4]

St. Augustine deserves also the credit of having accepted the existence of the *actually infinite* and to have recognized it as being, not a variable, but a constant. He recognized *all* finite positive integers as an infinity of that type.[5] On this point St. Augustine occupied a radically different and more advanced position than his forerunner, Origen of Alexandria, who took a decided stand against the actually infinite and supported his position by arguments which G. Cantor admits to be the profoundest ever advanced against the actually infinite.[6] In their completest form, these arguments were given many centuries later by the great Italian philosopher of the Middle Ages, Thomas Aquinas (1225(?)–1274).[7] Of importance to us is the nature of the continuum, particularly the linear continuum, as described by Aquinas. It was conceived as *potentially* divisible to infinity, since practically the divisions could not be carried out to infinity. There was, therefore, no minimum line. On the other hand, the point is not a constituent part of a line, since it does not possess the property of infinite divisi-

[1] Lucretius, *De rerum natura*, ed. I. Bernays, Leipzig, 1886, I, 615 ff.

[2] *De natura deorum*, Book III, IX.

[3] Epistola 88.

[4] St. Augustine, *De ordine*, II, VI, 18.

[5] S. Augustin, *De civitate Dei*, lib. XII, cap. 19. The chapter is quoted by G. Cantor in "Mitteilungen zur Lehre vom Transfiniten," *Zeitschr. f. Philosophie u. philosoph. Kritik*, Bd. 91, p. 81. Separatabdruck, p. 32.

[6] G. Cantor, *op. cit.*, Separatabdruck, p. 35.

[7] Thomas Aquinas, *Summa theol.*, I, q. 7 a. 4. Quoted by G. Cantor, *op. cit.*, Separatabdruck, p. 36.

bility that parts of a line possess, nor can the continuum be constructed out of points. However, a point by its motion has the capacity of generating a line.[1]

This concept of the continuum, as held by Aquinas, is a fair representation of the prevalent medieval scholastic views on this topic. It held a firm ascendancy over the ancient atomistic doctrine which assumed matter to be composed of very small, indivisible particles, possessing the properties of matter itself. No continuum superior to it was created before the nineteenth century.

In his commentaries on Aristotle's *Physics*, Aquinas[2] explains at some length the arguments of Zeno against motion as they are given by Aristotle. Aquinas shows a complete mastery of the subject as expounded by the Stagirite, but hardly presents any new points of view.[3]

Early English Writers. The earliest Englishman known to have written on continuity and infinity is Roger Bacon (1214(?)–1294), the seven-hundredth anniversary of whose birth was celebrated at Oxford in 1914. Bacon argued against the composition of the continuum of *indivisible* parts (different from points), by renewing the arguments presented by the Greeks and the early Arabs. Bacon held that the hypothesis of indivisible parts of uniform size would make the diagonal of a square commensurable with a side; if the ends of an indivisible part of a circle are connected by radii with the center of the circle, then the two radii would intercept an arc on a concentric circle of smaller radius. From this it would follow that the inner circle is of the same length as the outer circle. This is impossible. Bacon argued also against infinity. If time were infinite, it would follow that the part is equal to the whole—a deduction which he considered absurd. Similar arguments lead him to conclude that the world is finite.[4]

The views of Roger Bacon became known more widely through Duns Scotus (1265–1308), the theological and philosophical opponent of Thomas Aquinas. However, Scotus and Aquinas took the same ground in teaching that in the continuum there existed actual, indivisible points. Thereby it is not admitted that the continuum is made up of, or consists wholly of, points; the indivisible

[1] C. R. Wallner, in *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, pp. 29, 30, gives quotations from Thomas Aquinas, *Opuscula omnia*, 1562, o. 52, p. 369; o. 36, c. 2; o. 44, c. 1; o. 44, c. 2, p. 280.

[2] *Opera omnia*, Tom. II, Pars prima: *Sancti Thomae aqvinatis ex ordine praedicatorum quinti ecclesiae doctoris angelici praeclarissima commentaria in octo Physicorum Aristotelis libros. . . . Ad haec accessit Roberti Linconiensis in eosdem summa.* Parisiis, MDCLX, Lectio XI, pp. 233–237, 352.

[3] A summary of what is given in each of Aristotle's eight books on Physics is given by Robert Grosseteste (1175(?)–1253), bishop of Lincoln, whom Roger Bacon praises as a scientist of high rank. The esteem in which his writings were held appears from the fact that his summary of Aristotle is reproduced four centuries after his death in an edition of Aquinas. Of Zeno's arguments Grosseteste says (page 352): *Ad primum dicitur (sicut prius dictum est id dubitatione præcedenti) scilicet quòd continuum est infinitum secundum potentiam & tale potest transiri. Et sic patet ad secundam rationem. Ad tertium dicitur quòd Zeno dixit ipsum tempus componi ex instantibus, quod non est verum, ideò nec motus nec quies est in instanti, sed in tempore [sicut dixit Philosophus] ideò mobile non est spacio sibi aequali nisi tantum in instanti. Et cum dicitur, aut movetur, aut quiescit, negatur propositio, quia habet veritatem de es quod est in aliquo in quo aptum natum moveri aut quiescere pro tali mensura temporis.*

[4] See *Opera hactenus inedita Baconi*, Fasc. 1, Metaphysica. Edidit Robert Steele. London, p. 11; Jonas Cohn, *Geschichte des Unendlichkeitsproblems*, Leipzig, 1896, pp. 76, 77; K. Lasswitz, *op. cit.*, Vol. I, pp. 193, 195.

points might, for instance, be simply end points. These contentions are directed against the atomists. The arguments are wanting in explicitness and precision. What we said of Aquinas's commentary on Zeno applies also to Duns Scotus.[1] He gives detailed elaborations of Aristotle without offering new explanations of Zeno's puzzles. In place of Achilles and the tortoise he introduces the more familiar travelers, the horse and the ant. His commentaries are annotated by the Franciscan theologian Franciscus de Pitigianis of Arezzo in Italy, who wrote the latter part of the sixteenth century. This annotator expresses himself in favor of the admission of the actual infinity to explain the "Dichotomy" and the "Achilles," but fails to adequately elaborate the subject. Scholastic ideas on infinity and the continuum find expression in the writings of Bradwardine, the English *doctor profundus*. He says that five explanations have been given of the nature of the continuum.[2]

GROUPS OF SUBTRACTION AND DIVISION WITH RESPECT TO A MODULUS.

By G. A. MILLER, University of Illinois.

Certain kinds of groups of subtraction and division were explained by the present writer in two articles entitled: "Groups of the fundamental operations of arithmetic" and "Groups of subtraction and division." These articles were published respectively in the *Annals of Mathematics*, volume 6 (1905), page 41; and in the *Quarterly Journal of Mathematics*, volume 37 (1906), page 80. The present article is devoted to more elementary considerations, and has for its main object to exhibit interesting elementary relations between certain groups of subtraction and division, and the corresponding groups of addition and multiplication.

It is well known, and also evident, that the first $m - 1$ natural numbers together with zero constitute the cyclic group of order m with respect to addition when the sums are replaced by their least positive residues, or by zero, modulo m. That is, if in the series of numbers

$$0,\ 1,\ 2,\ \cdots,\ m-1$$

each number is replaced by itself increased by α, mod m, where $0 \gtreqless \alpha \gtreqless m - 1$, there results a certain substitution on these m numbers, and the totality of the distinct substitutions which can be constructed in this manner constitutes the cyclic group of order m. The order of the substitution corresponding to α is

[1] Duns Scoti, *Opera Omnia*, T. II: Joannis Duns Scoti Doctoris Subtilis, ordinis minorum, in VIII libros Physicorum Aristotelis Quæstiones, cum annotationibus R. P. F. Francisci Pitigiani arretini, etc. Lvgdvni, MDCXXXIX, Quaestio X, pp. 390–393.

[2] See Maximilian Curtze on the "Tractatus de continuo Bradwardini" in *Zeitschrift f. math. u. Phys.*, XIII Jahrg., Suppl., 1868, Leipzig, p. 88.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

IV.

By FLORIAN CAJORI, Colorado College.

4. EARLY DISCUSSIONS OF LIMITS: GREGORY ST. VINCENT, GALILEO, HOBBES.

Limits in the Fifteenth and Sixteenth Centuries. With the fifteenth century new mathematical ideas appear. These germs are found in Greek philosophy, but they failed to develop during the dark centuries. In the fifteenth century the German cardinal, Nicolaus Cusanus (1401–1465), considered variability without being able to apply it successfully; he advanced the notion of a limit, though unable to pass correctly to the limit; he entertained the notion of infinitesimals but was not able to use them in an infinitesimal calculus.[1] He held that rules developed for the finite lose their validity for the infinite—a statement which later thinkers have not always heeded sufficiently. A point moving with infinite velocity in a circle is each moment in every position on the circle; hence it is at rest.

During the century, or century and a half, after Cusanus, concepts of limits and processes involving the passing to the limit begin to appear in different parts of Europe, like flowers on a field in early spring. Perhaps first in time, in the development of ideas considered by Cusanus, is Giovanno B. Benedetti, a distinguished forerunner of Galileo, who brought out a publication in 1585 at Turin, Italy. As early as 1586, and again in 1608, Simon Stevin at Leyden exhibited the process of passing to the limit.[2] In 1604 the Italian mathematician, Luc Valerio, published at Rome a treatise, *De centrogravitatis*, which contains a remarkable approach to the modern idea of limits.[3] In Galileo's celebrated

[1] K. Lasswitz, *Geschichte der Atomistik*, Hamburg und Leipzig, 1. Bd., 1890, pp. 283, 284, 287, See also Max Simon, "Cusanus als Mathematiker," *Festschr. H. Weber*, Leipzig und Berlin, 1912. pp. 298–337.

[2] H. Bosmans, "Sur quelques exemples de la méthode des limites chez Simon Stevin," *Annales de la société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule.

[3] H. Bosmans, "Les démonstrations par l'analyse infinitésimale chez Luc Valerio," *Annales de la Société scientifique de Bruxelles*, T. 37, 1912–13, 2. fascicule; C. R. Wallner, "Ueber die Entstehung des Grenzbegriffes," *Bibliotheca mathematica*, 3. F., Bd. IV, 1903, p. 250,

discourses on mechanics and falling bodies (1638) there are frequent instances of limits. In the Netherlands again, Gregory St. Vincent, whose researches have, until recently, hardly received the recognition they deserve, was familiar with the writings of Luc Valerio, and himself contributed toward laying the foundations for the infinitesimal calculus. Similar studies bearing on the concept of a limit are due to Andreas Tacquet of Antwerp, and to John Wallis in his *Arithmetica infinitorum*, 1655, who were both familiar with the *Opus geometricum* of Gregory St. Vincent.[1]

We proceed now to a special mention of discussions of Zeno's arguments. Benedetti, whom we mentioned above, held that the flying arrow, thought of at a point in its path, does not cover a finite distance, but it differs from an arrow at rest by possessing the attribute of velocity which persists even in an infinitesimal time and space.[2] Direct reference to Zeno in a manner which exhibits reckless following of the great dialectician is found in Giuseppe Biancani of Bologna who about 1615 sought to establish the incommensurability of two lines by the consideration that a supposed common measure could not be applied to either line, because the measure must first be applied to half of it, and before that to half of that half, and so on to infinity, which is as impossible an operation as Zeno's "Dichotomy."[3]

Speculations of Galileo. Far more successful than earlier writers in the application of infinitesimals were Kepler and Cavalieri, but more important to us at present are the speculations of Galileo. Galileo approached the problem of infinite aggregates with a keenness of vision and an originality which was not equalled before the time of Dedekind and Georg Cantor. Galileo's dialogues on mechanics, *Discorsi e Dimostrazioni matematiche*, 1638, opens the "first day" with a discussion of divisibility and continuity of matter and space.[4] Salviati, who in general represents the author's own ideas, says, [5]"the infinite is inconceivable to us, as is the last indivisible." Simplicio, who in these dialogues is the spokesman of Aristotelian scholastic philosophy, remarks that "the infinity of points on a longer line must be greater than the infinity of points on a shorter one." Then come the remarkable words of Salviati:

"These difficulties arise because we with our finite minds discuss the infinite, attributing to the latter properties derived from the finite and limited. This, however, is not justifiable; for the attributes great, small and equal are not applicable to the infinite, since one cannot speak of greater, smaller, or equal infinities. . . . If now I ask how many squares are there, one can answer with truth, just as many as there are roots; for every square has a root,[6] every root has a square, no square has more than one root, no root more than one square. . . . I see no escape, except to say: the totality of numbers is infinite, the totality of squares is infinite, the totality of roots is infinite; the multitude of squares is not less than the multitude of numbers, neither is

[1] C. R. Wallner, *loc. cit.*, p. 257.

[3] K. Lasswitz, *op. cit.*, Vol. II, p. 17.

[3] J. C. Heilbronner, *Historia matheseos universæ*, Lipsiæ, 1742, p. 175.

[4] See a German translation in *Ostwald's Klassiker*, No. 11, pp. 24–37, also No. 24, p. 17; an English translation of the parts bearing on aggregates is given by E. Kasner in *Bulletin Am. Math. Soc.*, Vol. XI, 1904–5, pp. 499–501.

[5] *Ostwald's Klassiker*, No. 11, p. 29.

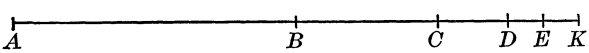
[6] Following the custom of his time, Galileo considers only one root of a positive number, namely the principal root.

the latter the greater; and, finally, the attributes equal, greater, and less are not applicable to infinite but solely to finite quantities."

We shall see that Galileo has been curiously misinterpreted by some writers, including Cauchy, as demonstrating here that an actual infinity has no existence. That there should be as many squares as there are integers altogether was taken as absurd; hence the existence of actual infinity was considered disproved. Galileo's skill in the use of the infinite in demonstrations is shown in the following passage on falling bodies:[1]

"If the velocity were proportional to the distance through which it has fallen or is to fall, then those distances would be passed over in equal times; thus, if the velocity with which a body overcomes four yards is to be double the velocity with which the first two yards were overcome, then the times needed for these two processes would be the same; but four yards can be overcome in the same time as two yards only in the case of instantaneous motion; we see on the contrary that the body needs time to fall, and that it needs less time for a fall of two yards than of four yards; hence it is not true that the velocities increase proportionally to the distance fallen."

Gregory St. Vincent. The most important discussion of Zeno given at this time is that by Gregory St. Vincent, in his *Opus geometricum quadraturæ circuli et sectionum coni*, published at Antwerp in 1647, but written apparently twenty-five years earlier. It is a massive volume of 1400 pages. Influenced in his geometrical researches by the medieval scholastic concept of the continuum, according to which a line divided repeatedly is not reduced to indivisible elements as taught by the atomists, but admits of being subdivided *ad infinitum*, Gregory St. Vincent took a step different from that of Archimedes. While, in his proofs, Archimedes kept on dividing, only until a certain degree of smallness was reached, St. Vincent permitted the subdivisions to continue *ad infinitum*. Using unlimited section in geometry he introduced a geometric series that was truly an *infinite* series.[2]



This much had been accomplished by at least one writer before him,[3] but, so far as now known, he is the first to apply the infinite geometric progression to the study of the "Achilles." Taking a definite line segment AK he divides it at B in a given ratio, then he divides BK in the same ratio at C, and so on. The segments AB, BC, CD, . . . form an infinite geometric progression. The points C, D, E . . . lie, all of them, between A and K; they approach K as near as we please, but (in accordance with scholastic philosophy) never reach it. As Gregory conceives this matter, K is an obstacle, so to speak, against the further advance of the series of points A, B, C, . . . , similar to a rigid wall. "Terminus progressionis est seriei finis, ad quem nulla progressio pertinget, *licet* in infinitum continuetur; sed quovis interuallo dato proprius ad eum *accedere* poterit." By "series" is meant the segment AK, by "progressio," the segments AB, AC, ...

[1] *Ostwald's Klassiker*, No. 24, p. 17.

[2] Gregory St. Vincent, *Opus geometricum*, T. 1, pp. 51–56, 95–97; for our knowledge of this part of the book we are dependent entirely upon C. R. Wallner's account in the *Bibliotheca mathematica*, 3. F., Vol. IV, 1903, pp. 251–255.

[3] See H. Wieleitner in *Bibliotheca mathematica*, 3. F., Vol. 14, 1914, pp. 150–168.

Gregory states his conclusion thus: "Dico magnitudinem AK aequalem esse toti progressioni magnitudinum continue proportionalium, rationis AB ad BC in infinitum continuatae; siue quod idem est, rationis AB ad BC in infinitum continuatae terminum esse K." Considering the "Achilles" in this connection, he associates this paradox on motion for the first time definitely with the summation of an infinite series. Moreover, Gregory St. Vincent is the first writer known to us who states the exact time and place of overtaking the tortoise. So far as we are able to ascertain, Gregory was not troubled, in explaining the "Achilles," by the fact that in his theory, the variable does not *reach* its limit. Nor, apparently, did this matter trouble his readers. His mode of solving the problem appealed to many. We shall see that Leibniz makes special reference to it. Over a century after Gregory's publication, Saverien refers in his dictionary[1] to the "Achilles," "dont Gregoire de Saint Vincent a fait voir la fausseté." Formey gave Gregory St. Vincent's explanation in the article "Mouvement" in Diderot's *Encyclopédie* (1754), later reprinted in the *Encyclopédie méthodique*, and in 1800 translated at Padova into the Italian language. The definition of a limit as given in the *Encyclopédie méthodique* does not allow the variable to surpass its limit but places no obstacle in the way of its reaching its limit.

Descartes, De Morgan and Others. Descartes at one time discussed the "Achilles." His treatment is much like that of Gregory St. Vincent. It is given in a letter of July, 1646, to Clerselier.[2] He lets Achilles, or in his place a horse, be, at the start, 10 leagues behind the tortoise, but moving ten times more rapidly than the latter. The real difficulty of the paradox he does not touch, for he says:

> "L'Achille de Zenon ne sera pas difficile à soudre, si on prend garde que, si à la dixiéme partie de quelque quantité on adioute la dixiéme de cette dixiéme, qui est une centiéme, & encore la dixiéme de cette derniere, qui n'est qu'une milliesme de la premiere, & ainsi à l'infini, toutes ces dixiémes jointes ensemble, quoy qu'elles soient suposées réellement infinies, ne composent toutesfois qu' une quantité finie, sçauvoir une neusiéme de la premiere quantité . . . Et la caption est en ce qu'on imagine que cette neusiéme partie d'une lieuë est une quantité infinie, à cause qu'on la divise par son imagination en des parties infinies."

Descartes looked upon the actually infinite as mysterious, but not impossible or absurd. He seemed to accept it in the abstract, but deny it in the concrete. At this time and even earlier (see the foregoing extracts from Galileo) there was talk about the finitude of the human mind and its consequent inability to conceive the infinite. This was ridiculed by De Morgan. He claimed that if the human mind is limited, we tacitly postulate the "unknowable"; moreover, even if the human mind were finite, there is no more reason against its conceiving the infinite than there is for a mind to be blue in order to conceive of a pair of blue eyes. Or, as De Morgan puts it in another place, the argument amounts to this, "who drives fat oxen should himself be fat." From Descartes to Hamilton,

[1] Saverien, *Dictionnaire universel de mathèmatique et de physique*, Paris, 1753, Art. "Mouvement."

[2] *Oeuvres de Descartes* par Charles Adam et Paul Tannery, T. IV, pp. 445–447.

says De Morgan,[1] this doctrine is accepted by many minds. But its genesis is found, as we have stated, long before Descartes.

A wholly different, but no more satisfactory explanation of "Achilles" comes from another Frenchman of that time, Pierre Gassendi, the physicist. In his view Zeno's proofs need no refutation, if with Epicurus one assumes not points but atoms. A difficulty seems to arise from differences in velocity of motion, for in the same time that a body moves over the physically indivisible, the more rapid body must travel over several indivisibles. In his opinion this difficulty may perhaps be overcome by conceiving motion as discontinuous, and slower motion as a mixture of rest and motion. To the senses motion would still seem continuous.[2] To those who experienced difficulties in accepting the existence of indivisible atoms, the capuchin, Casimir of Toulouse, offers an easy solution by reminding that angels had extension, yet were physically indivisible.[3]

It is worthy of note that John Dee, the famous astrologer who wrote an elaborate mathematical preface to Billingsley's edition of Euclid (1570), departs from the contention that two lines containing the same number of parts must be of equal length. He says:

"Our least Magnitudes can be divided into so many partes as the greatest. As, a Line of an inch long (with vs) may be divided into as many partes, as may the diameter of the whole world, from East to West: or any way extended."

Discussion of Thomas Hobbes. The earliest British writer, after Duns Scotus, to take up explicitly Zeno's arguments is the philosopher, Thomas Hobbes (1588–1679). In 1655 he wrote:[4]

". . . the force of that famous argument of Zeno against motion, consisted in this proposition, *whatsoever may be divided into parts, infinite in number, the same is infinite;* which he without doubt, thought to be true, yet nevertheless is false. For to be divided into infinite parts, is nothing else but to be divided into as many parts as any man will. But it is not necessary that a line should have parts infinite in number, or be infinite, because I can divide and subdivide it as often as I please; for how many parts soever I make, yet their number is finite; because he that says parts, simply, without adding how many, does not limit any number, but leaves it to the determination of the hearer, therefore we say commonly, a line may be divided infinitely; which cannot be true in any other sense."

With Hobbes, *infinite* is synonymous with *indefinite.* He takes an agnostic attitude toward problems of infinity:

"But when no more is said than this, *number is infinite,* it is to be understood as if it were said, this name *number* is an *indefinite* name. . . . And, therefore, that which is commonly said, that space and time may be divided infinitely, is not to be so understood, as if there might be any infinite or eternal division; but rather to be taken in this sense, *whatever is divided is divided into such parts as may again be divided.* . . . Who can commend him that demonstrates thus? 'If the world be eternal, then an infinite number of days, or other measures of time, preceded the birth of Abraham. But the birth of Abraham preceded the birth of Isaac; and therefore one

[1] A. De Morgan, "On Infinity; and on the Sign of Equality," in Trans. of the Cambridge Philosoph. Society, Vol. XI, p. 157, Cambridge, 1871 [read May 16, 1864].

[2] Gassendi, *Opera omnia,* 1658, I, p. 300a. An. I, p. 239; Lasswitz, *op. cit.,* Vol. II, p. 150.

[3] Lasswitz, *op. cit.,* Vol. II, p. 494.

[4] *The English Works of Thomas Hobbes,* Vol. I, London, 1839, pp. 63, 64, 413. Hobbes refers to Zeno's arguments also in his Latin works. See Thomae Hobbes, *Opera philosophica,* Vol. V, Londini, 1845, pp. 207–213.

infinite is greater than another infinite, or one eternal than another eternal; which,' he says, 'is absurd.' This demonstration is like his, who from this, that the number of even numbers is infinite, would conclude that there are as many even numbers as there are numbers simply, that is to say, the even numbers are as many as all the even and odd together. They which in this manner take away eternity from the world, do they not by the same means take away eternity from the creator of the world? . . . And the men that reason thus absurdly are not idiots, but, which makes this absurdity unpardonable, geometri r as, and such as take upon them to be judges, impertinent, but severe judges of other men's demonstrations."

The reference to odd and even numbers doubtless arose from his contact with Galilean thought. While sojourning on the Continent, he had gone to see Galileo, then a prisoner. Hobbes thought he had effected the duplication of the cube and the squaring of the circle. On this matter he became involved in a heated controversy with the algebraist, John Wallis. The aged Hobbes was no match against young Wallis on mathematical questions. When the mathematical works of Wallis were being brought out, Wallis refused to allow his controversial matter against Hobbes to be incorporated in them.[1] Whether the whole is greater than a part was an issue touched upon during this dispute. Hobbes said to Wallis: "All this arguing of infinities is but the ambition of school boys." It cannot be said that Hobbes made any real contribution to a deeper understanding of the "Achilles" or any of Zeno's other arguments on motion. His objection to the dictum, "whatever may be divided into parts infinite in number, the same is infinite," is no new contribution; Aristotle had advanced that far. How Achilles caught the turtle is beyond comprehension through our sensual imagination; Hobbes nowhere explains this inability. However, he does touch upon the concept of a limit in his controversy with Wallis. Hobbes charged that some of the principles of the professors are "void of sense"; one of those principles being, "that a quantity may grow less and less eternally, so as at last to be equal to another quantity; or, which is all one, that there is a last in eternity."[2]

A GENERAL FORMULA FOR THE VALUATION OF SECURITIES.[3]

By JAMES W. GLOVER, University of Michigan.

The object of this paper is to derive a formula for the valuation of a very general type of securities. The security is redeemed in r equal installments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m installments, and the security is purchased to realize the investor a nominal rate of interest j with frequency of conversion m.

[1] A full account of the controversy between Hobbes and Wallis is given in Croom Robertson's *Hobbes*, pp. 167–185.

[2] *The English Works of Thomas Hobbes*, Vol. 7, p. 186.

[3] Read before the Chicago Section of the American Mathematical Society, April, 1912. Those unfamiliar with the notation and functions employed in the theory of compound interest may consult *Text-Book of the Institute of Actuaries*, Part I, by Ralph Todhunter; *The Mathematical Theory of Investment*, by Ernest B. Skinner; *Bulletin of the Department of Agriculture*, No. 136, on Highway Bonds, by Laurence I. Hewes and James W. Glover.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

V.

By FLORIAN CAJORI, Colorado College.

5. BAYLE, LEIBNIZ AND OTHER CONTINENTAL WRITERS.

Speculations by Pierre Bayle. An unusually elaborate, detailed, and critical discussion of Zeno's arguments and Aristotle's refutations is given by Pierre Bayle in the article "Zenon d'Elée," printed in his *Dictionnaire historique et critique*, 1696. Our quotations are from the English translation of the dictionary, brought out in 1710 in London. Bayle was a noted French skeptical philosopher; his article on Zeno has been widely quoted. He begins with observations on the "Arrow." Every one admits, he says, that two bodies cannot be in the same place at the same time; that "two parts of Time cannot exist together" is a theorem which "requires a little more reflection in order to apprehend it." Bayle continues:

"I will render it more obvious by an Instance. I say then that what suits Monday and Tuesday with respect to succession, suits every portion of Time whatsoever. Since then it is impossible for Monday and Tuesday to exist together, and that of necessity Monday must cease to be before Tuesday begins to be, there is no part of time whatsoever, which can co-exist with another; each must exist alone; each must begin to be, when the precedent ceaseth to be; and each must cease to be before the following can begin to exist. From whence it follows, that Time is not divisible *in infinitum*, and that the successive duration of things is composed of Moments, properly so called, each of which is simple and indivisible, perfectly distinct from time past and future, and contains no more than the present time. Those who deny this Consequence, must be given up to their Stupidity, or their want of Sincerity, or the insurmountable power of their prejudices. But if you once grant that the present time is indivisible, you will be unavoidably obliged to admit Zeno's Objection. You cannot find an instant when the Arrow leaves its place; for if you find one, it will be at the same time in that place, and yet not there. Aristotle contents himself with answering, that Zeno very falsely supposes the indivisibility of Moments."

This is Bayle's singular argument by which he tries to show that time is composed of a finite number of indivisible parts, and that, in consequence of this property of time, the difficulties of Zeno's argument that the arrow does not

move, are satisfactorily removed. The heat displayed by Bayle at the close leads one to surmise that he himself had encountered opposition to his own explanation.

While Bayle denied the infinite divisibility of time, he admitted the infinite divisibility of space. He gives Zeno's argument in the "Dichotomy" and then says against Aristotle:

"To this Aristotle makes a wretched Answer: He sayth that a foot of matter being no otherwise infinite than in Power, may very well be run through in a finite time. . . . You have here two Particulars: 1. That each part of Time is divisible *in infinitum;* which is invincibly refuted above. 2. That a Body is only Infinite in Power. Which signifies that the Infinity of a Foot of Matter, consists in that it may be divided without end into smaller parts, but not in its being actually susceptible of that division. To urge this is to impose on the World; for if Matter is divisible *in infinitum,* it actually contains an infinite number of parts, and is not therefore an infinite in Power, but an Infinite which really and actually exists. . . . Don't Aristotle and his Followers assert, that an Hour contains an infinity of parts? Wherefore when it is past, it must be owned that an infinity of parts did actually exist one after another. Is this a virtual, and not an actual infinity? Let us then say that this Distinction is null, and that Zeno's Observation remains in full force. . . . Let us content our selves in this place with observing, that the subterfuge of the Infinity of the Parts of Time is null; for if there were in an Hour an infinity of Parts, it could never either begin or end."

What Bayle says in criticism of Aristotle has considerable force; what he says by way of a constructive argument—accepting the infinite divisibility of space, but denying it for time—fails to convince. Bayle says but little on "Achilles" but makes extensive, though uninteresting, remarks on the "Stade."

After this preliminary survey of Zeno and Aristotle, Bayle starts out anew, stating that Zeno very probably alleged other arguments which were perhaps the same as those he himself was about to mention, "some of which oppose the existence of Extension, and seem much stronger than all the Reasons which the Cartesians can allege." He continues:

"I am apt to think that those who would revive Zeno's Opinion, ought to argue thus. 1. There is no Extension, therefore there is no Motion. The Consequence is good, for what hath no Extension takes up no room, and what takes up no room cannot possibly pass from one place to another, nor consequently move. This is incontestable: The difficulty is then to prove that there is no Extension. Zeno might have argued thus. Extension cannot be composed of Mathematical Points, Atoms, or Parts divisible *in infinitum;* therefore its Existence is impossible. . . . A few Words shall suffice as to Mathematical Points; for . . . several nullities of Extension joined together will never make an Extension. Consult the first Body of Scholastical Philosophy that comes to hand, and you will find the most convincing Reasons in the World, supported by many Geometrical Demonstrations against the existence of these Points."

After this appeal to scholastic philosophy on the impossibility of a continuum made up of points, Bayle argues against the extended indivisible particles, called Epicurean atoms; their indivisibility is "chimerical," as each atom has a right side and a left side. Divisibility *in infinitum* leads him to the following caustic observations:

"The divisibility to *in infinitum* is an Hypothesis embraced by Aristotle, and almost all the Professors of Philosophy, in all Universities for several Ages. Not that they comprehend it, or can answer the Objections it is liable to; but because having clearly apprehended the impossibility of either Mathematical or Physical Points, they found no other side but this to take. Besides, this Hypothesis affords great Conveniences: For when their Distinctions are exhausted, without being able to render this Doctrine comprehensible, they shelter themselves in the nature of the Subject, and allege, that our Understandings being limited, none ought to be surprized that they

cannot resolve what relates to Infinity, and that it is essential to such a Continuity to be liable to such Difficulties as are insurmountable by Humane Nature. . . . To be convinced of their weakness, it is enough to remember that the strongest of them [the three hypotheses, points, atoms, parts infinitely divisible], that which best disputes the ground, is the Hypothesis of the divisibility to *in infinitum*. The Schoolmen have armed it from head to foot with all the Distinctions which their great leisure would allow them to invent: But all this only serves to afford their Scholars Matter for talk upon a public Disputation, that their Relations may not suffer the Disgrace of seeing them mute. And accordingly a Father or a Brother go away better satisfy'd, when the Scholar distinguishes betwixt a *Categorematical Infinite*, and a *Syncategorematical* one . . . than if he should answer nothing. It was therefore necessary for the Professors to invent a sorte of Jargon; but all the pains which they have taken, will never be able to obscure this Notion which is as clear and evident as the Sun: *An infinite number of Parts of Extension, each of which is extended, and distinct from all others, as well with respect to its Entity, as the room is taken up, cannot be contained in a Space One hundred Millions of times less than the Hundred Thousandth part of a Barly Corn.*"

That Bayle was thinking more of external realities than pure concepts of the human mind appears from the following quotation:

"What the Mathematicians acknowledge with respect to Lines and Superficies, with which they demonstrate so many excellent things, must be owned to be true of Bodies. They honestly own that length and breadth without depth, are things which cannot possibly exist anywhere but in our Imagination. As much may be said of the three Dimensions. They cannot find any room any where besides in our Minds; nor can they exist any other way than Notionally. . . ."

The demonstrations which have been given to show infinite divisibility are interpreted by Bayle as really proving that extension does not exist.

"In the 1. place I observe that some of these Demonstrations are made use of against those who affirm that Matter is composed of Mathematical Points. It is objected to them, that the Sides of a Square would be equal to the Diagonal, and that amongst concentrical Circles, the least would be equal to the largest. This Consequence is proved by making it appear that the right Lines which may be drawn from one of the sides of a Square to another will fill the Diagonal, and that all the right Lines which may be drawn from the Circumference of the largest Circle, will find room in the smallest Circumference. . . . In the 2. place I affirm that it being very true that if Circles did exist, as many right lines might be drawn from the Circumference, to the Center, as there are parts in the Circumference, it follows that the existence of a Circle is impossible. I assure myself that it will be allowed me that every Being which cannot exist, without containing properties which cannot exist, is impossible: But a round Extension cannot exist, without having a Center, in which as many right Lines as there are parts in the Circumference meet; and it is certain that such a Center cannot exist; It must then be owned that the Existence of this round Extension is impossible."

Further indication of the difficulties encountered in the effort to construct a non-contradictory continuum is exhibited in the following passage by Bayle:

". . . a Body in motion, rolling in a sloping Table, could never fall off the said Table; for before it falls, it must of necessity touch the last part of the Table. And how will it touch that, since all those parts which you will take for the last, contain an Infinity of parts, and an infinite Number hath no part which can be last? This Objection obliged some Scholastic Philosophers to suppose that Nature hath intermixed Mathematical Points with the parts divisible *in infinitum*, to the end that they may serve to connect them, and compose the extremities of Bodies."

And in conclusion Bayle says:

"Thus . . . we may suppose our Zeno of Elea to have opposed Motion. I will not aver that his Reasons persuaded him that nothing moved. . . . If I should judge of him by my self, I should affirm that he as well as others believed the motion of Extension; for tho' I find my self very incapable of solving all the difficulties which we have just now seen . . . I am persuaded that the exposition of these Arguments may be of great use with respect to Religion. . . . The advantage which may be drawn from these speculations is not merely to acquire this sort of Knowledge, which in itself is very barren; but to learn to know the bounds of our understanding."

Bayle tells also of the Sophist Diodorus who lectured against the existence of motion.[1] Having put his shoulder out of joint, he went to a physician to have it set. How? said the Doctor. Your Shoulder dislocated! That cannot be; for, if it moved, it did so either in the place where it was, or in the place where it was not. But it did not move either in the place where it was, or in the place where it was not, for it could neither act nor suffer in the place where it was not.

Bayle's article, though by far the fullest discussion of Zeno given up to that time, is not an illuminating contribution. It is prepared without much coordination. At first he refers approvingly to the infinite divisibility of space, later he speaks sneeringly of the advocates of that view. No mention is made of the important work of Gregory St. Vincent. Bayle's general attitude in his article is that of a skeptic.

Views of Leibniz. About the time when Bayle prepared his article, Leibniz touched upon the "Achilles" in his correspondence with the French philosopher Foucher. We may premise that lack of space prevents us from attempting a systematic exposition of Leibniz's infinitesimals and their use in his calculus. Like Newton, Leibniz changed his point of view on some fundamental concepts of mathematics, as the years rolled on. This change of base cannot be brought as a charge against either of them. They were encountering most subtle problems, in many different fields of inquiry; their ideas were in a state of flux. It is owing to this circumstance, as Vivanti has pointed out,[2] that different authors have attributed to Leibniz opposite views and each was able to fortify his contention by direct quotation. Thus Wolf, Achard, Gerdil, and Mansion denied that Leibniz admitted the existence of infinitesimals different from zero; Grandi claimed the opposite; Cohen and Lasswitz attributed to him the concept of the intensive infinitesimal, that is, an infinitesimal considered as the generator of finite magnitude, though itself without magnitude. In March, 1693, Foucher wrote Leibniz a letter in which he asks for information, how Leibniz could consistently admit divisibles and also indivisibles, and dwells upon the difficulties offered by the various alternatives: indivisible instants corresponding to divisible dots (points), or divisible instants corresponding to indivisible dots, or divisible instants corresponding to divisible dots. In the case of the last alternative, "on ne pourra resoudre la difficulté des Sceptiques, ni montrer comment Achille doit aller plus vite qu'une tortuë."[3] To this Leibniz replies[4] that twenty years previously he had written two discourses on motion which may contain some things of value but which contain passages on which he considers himself now better informed, "et entre autres, je m'explique tout autrement aujourdhuy

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[1] Sextus Empiricus, lib. 2, c. 22.

[2] Giulio Vivanti, "Il concetto d'infinitesimo e la sua applicazione alla matematica. Saggio storico," *Giornale di matematiche di Battaglini*, Vol. 38 e 39, Estratto, Napoli, 1901, p. 11. This research is a most valuable one, containing extensive quotations from a large number of original sources many of which are difficult of access.

[3] *Die philosophischen Schriften von Gottfried Wilhelm Leibniz*, herausgegeben v. C. I. Gerhardt, Bd. I, Berlin, 1875, p. 411.

[4] *Loc. cit.*, p. 415.

sur les indivisibles. C'estoit l'essay d'un jeune homme qui n'avoit pas ancor approfondi les mathématiques." Later passages in this letter are of interest in the light of recent concepts of the atom, as well as in the light of the modern theory of the continuum:

"Quand aux *indivisibles*, lorsqu'ou entend por là les simples extremités du temps ou de la ligne, ou n'y sçauroit concevoir de nouvelles extremités ny des parties actuelles ny potentielles. Ainsi les points ne sont ny gros ny petits, et il ne faut point de saut pour les passer. Cependant le continu, quoy qu'il ait partout de tels indivisibles, n'en est point composé, comme il semble que les objections de Sceptiques le supposent, qui, à mon avis, n'ont rien d'insurmontable, comme on trouvera en les redigeant en forme. Le père Gregoire de S. Vincent a fort bien montré par le calcul même de la divisibilité à infini, l'endroit où Achille doit attraper la tortue qui le devance, selon la proportion des vitesses. Ainsi la Geometrie sert à dissiper ces difficultés apparentes. Je suis tellement pour l'*infini actuel*, qu'au lieu d'admettre que la nature l'abhorre, comme l'on dit, vulgairement, je tiens qu'elle l'affecte partout, pour mieux marquer les perfections de son auteur. Ainsi je crois qu'il n'y a aucune partie de la matiere qui ne soit, je ne dis pas divisible, mais actuellement divisée, et par consequent, la moindre particelle doit estre considerée comme monde plein d'une infinité de creatures differentes."

It is interesting to observe that Leibniz's comment on Gregory St. Vincent's explanation of the "Achilles" was favorable, as were all comments of that time with which we are familiar. It was a time when a circle was quite generally looked upon as a polygon with an infinite number of sides; hence, no ultra refinements should be expected. Worthy of notice is a passage in a letter of Leibniz[1] to John Bernoulli I, dated Aug. 22, 1698, in which Leibniz declares that Burcher de Volder, and Gregory St. Vincent before him, rejected the axiom that the whole is greater than its part when it applied to infinity. Volder was a man of prominence in the Netherlands, as appears from the fact that he was selected to edit the works of Huygens. Leibniz could not agree with the Dutch scientists and called their views absurd. It is interesting to see how this idea of the whole not being greater than certain of its parts, so clearly brought out by Galileo for infinite aggregates, every now and then forced itself upon the attention of men pondering on the subject of infinity. The Spaniard, Juan Andrès,[2] quotes with approval from Christian Wolff's widely used book, *Elementa matheseos universae* (Arth. num. 86), a proof of the theorem that "the whole is greater than its part." On the other hand, the opposite view, held in the seventeenth century (as we have seen) by Galileo and Volder, found utterance in the eighteenth century in a book by Johann Schultz.[3]

Some Eighteenth Century Discussion. While considerable discussion took place on Zeno's arguments in the seventeenth century, by writers like Biancani, Gregory St. Vincent, Peter Bayle, Descartes, and Leibniz, comparatively little was said on this subject during the eighteenth century. There was tremendous activity during the eighteenth century in the fuller development of the differential and integral calculus and its applications. With it came a more systematic

[1] Got. Gul. Leibnitii et Johan. Bernoullii *Commercium Philosophicum et mathematicum*, T. 1, Lausannæ et Genevæ, 1745, pp. 389, 397.

[2] Juan Andrès, *De studiis philosophicis et matematicis*, Matriti, 1789.

[3] J. Schultz, *Versuch einer genauen Theorie des Unendlichen*, Königsberg und Leipzig, 1778, p. 87.

development of the theory of limits, but that development was not such as to really throw much new light upon infinite divisibility or the ability of variables to reach their limits. Much might be said on discussions of the infinite, but we shall confine our attention to views that bear more directly upon the subject of our inquiry.

The Italian philosopher and philologist, Jacopo Facciolati, of Padua, wrote upon the "Achilles" in *Acroases dialecticae,* Venetiis, 1750. As reported by Hoffbauer,[1] Facciolati makes an assumption, in accordance with which the tortoise is never caught, though Zeno's alleged contention that the swifter cannot catch the slower is not established thereby. Suppose $a, b, c, d, \ldots$ are points on a line, such that the distance bc is one-tenth the preceding distance ab, etc. The extra assumption was to the effect that both Achilles and the tortoise made stops at the points $a, b, c, \ldots$ so that the time of transit from one letter to the next did not fall below a certain minimum. This solution of the puzzle can hardly be ranked as a real advance. More searching was Father Gerdil (1718–1802) of Turin who ranked high as a professor and philosopher, and was finally given a cardinal's hat. In his article, *De l'infini absolu consideré dans la Grandeur,*[2] he quotes from an article by the French professor of mathematics and philosophy, l'abbé Deidier (1696–about 1746), who said that Zeno's conclusion is absurd, except on two suppositions: the first is that Achilles took an infinite number of steps to cover the first league, in which case he never reached his goal; the second is that when he passed $\frac{1}{10}$ of the previous distance, his steps also became ten times shorter, so that he could not reach the tortoise. As both of these suppositions are ridiculous and impossible, it follows that Zeno's argument is a mere sophism. If some one objects by saying that Achilles must travel $\frac{1}{9}$ of a league, which he cannot do since he has to pass through an infinite progression $\frac{1}{10}, \frac{1}{100}, \ldots$, I reply that this is a sophism as simple as the first, for Achilles continually travels at a uniform rate.

Father Gerdil endorses abbé Deidier's views. He himself points out that, if the tortoise has at starting the lead of 1 league, it travels a distance x before it is caught, where x is determined by $10x = 1 + x$. His mode of solving the "Achilles" consists in avoiding the summation of an infinite progression by addition of its terms, and in determining by one stroke the value represented by that progression. An infinite progression has no last term, yet says he, the number of terms does not constitute an actual infinity. His argument against the possibility of an actual infinity carried great weight with Cauchy.[3]

Passing to Germany we meet first with a philosophical publication by Johann

[1] J. S. Ersch und J. G. Gruber, Allg. Encyclopädie der Wissensch. u. Künste, Leipzig, 1818, Art. "Achilles."

[2] *Mélanges de philosophie et de mathématique de la Société Royale de Turin,* 1760–1761, Suppl., p. 1. Georg Cantor gives also the following article by Gerdil: "Essai d'une démonstration mathématique contre l'existence éternelle de la matière et du mouvement, déduite de l'impossibilité demontrée d'une suite actuellement infinie de termes, soit permanents, soit successifs," *Opere edite et inedite del cardinale Giacinto Sigismondo Gerdil,* T. IV, p. 261, Rome, 1806.

[3] See Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuale Unendliche," *Zeitsch. f. Philos. u. Philos. Kritik,* Bd. 88, p. 224.

Gottlieb Waldin,[1] professor of mathematics at Marburg, who declares Zeno's proofs invalid, because Zeno assumes the existence of motion, the very thing in dispute.

THE IDENTICAL RELATIONS BETWEEN THE DIRECTION COSINES OF ONE OBLIQUE COORDINATE SYSTEM REFERRED TO ANOTHER OBLIQUE SYSTEM.

By HENRY D. THOMPSON.

I. *Introduction.*

If $Oxyz$, $Ox'y'z'$ are any two oblique coordinate systems, then connecting the cosines of the fifteen angles between the six lines there are identical relations of the fourth degree which are simple and which are the exact counterpart of the well known twenty-two relations between two orthogonal systems. Relations for the oblique case have been given; for example, by Grunert[2] and by Sturm,[3] but some of these are of a degree higher than the fourth, and they are not always so easy to use as the orthogonal relations. In the *American Journal of Mathematics*, Vol. XXXV, p. 427, it has been proved that the so called Lamé and Gauss equations in the theory of surfaces are special cases of equations holding for oblique triple systems of surfaces. In the same way, in all the cases tried, it has been found that, by the use of the identical relation between two sets of four directions in space, all the usual orthogonal relations can be generalized and the corresponding formulas for the oblique cases can be obtained. This method will be employed here to obtain also the simple relations between oblique coordinate systems.

Let $Ox_0y_0z_0$ be an orthogonal system. Define the (direction) cosines of the angles between the various lines of $Oxyz$, $Ox'y'z'$, $Ox_0y_0z_0$ by the following schemes, $c_{ii} = 1$, $c_{ij} = c_{ji}$, $c_{ii}' = 1$, $c_{ij}' = c_{ji}'$,

$$\begin{array}{c|ccc} C & x & y & z \\ \hline x & c_{11} & c_{12} & c_{13} \\ y & c_{21} & c_{22} & c_{23} \\ z & c_{31} & c_{32} & c_{33} \end{array} \qquad \begin{array}{c|ccc} C' & x' & y' & z' \\ \hline x' & c_{11}' & c_{12}' & c_{13}' \\ y' & c_{21}' & c_{22}' & c_{23}' \\ z' & c_{31}' & c_{32}' & c_{33}' \end{array} \qquad \begin{array}{c|ccc} D & x' & y' & z' \\ \hline x & l_1 & m_1 & n_1 \\ y & l_2 & m_2 & n_2 \\ z & l_3 & m_3 & n_3 \end{array} \qquad \begin{array}{c|ccc} A & x_0 & y_0 & z_0 \\ \hline x & \lambda_1 & \mu_1 & \nu_1 \\ y & \lambda_2 & \mu_2 & \nu_2 \\ z & \lambda_3 & \mu_3 & \nu_3 \end{array} \qquad \begin{array}{c|ccc} A' & x_0 & y_0 & z_0, \\ \hline x' & \lambda_1' & \mu_1' & \nu_1' \\ y' & \lambda_2' & \mu_2' & \nu_2' \\ z' & \lambda_3' & \mu_3' & \nu_3' \end{array}$$

Call the corresponding five determinants C, C', D, A, A', and represent each cofactor in these determinants by the capital letter and the subscripts of the corresponding element. Since $\lambda_i^2 + \mu_i^2 + \nu_i^2 = c_{ii}$, $\lambda_i\lambda_j + \mu_i\mu_j + \nu_i\nu_j = c_{ij}$, etc., $i = 1, 2, 3$; $j = 1, 2, 3$; direct multiplication gives that $A^2 = C$, $A'^2 = C'$, and $A \cdot A' = D$, or $D = C^{\frac{1}{2}}C'^{\frac{1}{2}}$.

In the orthogonal case, only the elements of D appear, and in accordance with

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[1] J. G. Waldin, *Erste Gründe der allgemeinen und besondern Vernunftlehre*, Marburg, 1782, p. 26. Our information about Waldin is drawn from a history of Zeno's arguments by Eduard Wellmann, entitled "Zenos Beweise gegen die Bewegung und ihre Widerlegungen," in *Programm des Friedrichs-Gymnasiums zu Frankfurt A. O., für das Schuljahr* 1869–1870. Frankfurt A. O. 1870, p. 14.

[2] *Arch. d. Math.*, 34, p. 142 and fol.

[3] *Arch. d. Math.*, 3te R., 22, p. 327.

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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION.

By FLORIAN CAJORI, Colorado College.

VI.

6. NEWTON, BERKELEY, JURIN, ROBINS AND OTHERS.

Whether certain variables can reach their limits or not is the vital issue in the "Achilles." For that reason Newton's statements on this point are of interest:

". . . those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits toward which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished in *infinitum*."[1]

That Newton let his variables reach their limits appears even more clearly in the following passage:

"Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."[2]

Other passages in the first book of the *Principia* allow variables to reach their limits. While Newton's exposition is not as explicit as one might wish, nor free from objection, he deserves the credit of perceiving that variables may reach their limits and that variables arising in mechanics are usually of such a nature that they do reach their limits.

As is well known, the foundations of the calculus were severely attacked by Bishop Berkeley. His first published statement on this subject appears in his *Alciphron, or the Minute Philosopher* (1732), penned while he and his wife were sojourning at Newport, Rhode Island. He says that mathematical science falls short of "those clear and distinct ideas" which many "expect and insist upon in the mysteries of religion. . . . Such are those which have sprung up

[1] Newton's *Principia*, Book I, Section I, last scholium.

[2] Newton's *Principia*, Book I, Section I, Lemma I.

in geometry about the nature of the angle of contact, the doctrine of proportions, of indivisibles, infinitesimals, and divers other points." He argues that, just as mathematics, though involved in obscurities, is esteemed excellent and useful, so articles of Christian faith should be accepted as none the less true and excellent, because they afford matter of controversy. These ideas Berkeley elaborates more fully in his *Analyst* (1734) and the *Defence of Free-thinking in Mathematics* (1735). These attacks on the foundations of the calculus deserve mention in this history, even though no reference is made to Zeno's arguments. Berkeley was familiar with Zeno's arguments, for in an earlier essay he refers to them twice, though without critical comment.[1]

The *Analyst* is a discourse addressed to an unnamed infidel mathematician, said to have been Dr. Halley. Berkeley's lengthy discourse dwells mainly on two points: (1) The conception of fluxions is unintelligible, since they are the ratios of quantities that have no magnitudes, (2) the derivation of the fluxion of x^n rests on a violation of an axiomatic canon of sound reasoning. Newton did not take the fluxions infinitely small, but he originally took the *moments of fluxions* to be infinitely small. Later he discarded the infinitely little and explained fluxions by his theory of prime and ultimate ratios. Berkeley argued with great acuteness against the infinitely small. As his arguments do not directly apply to the "Dichotomy" and the "Achilles" we shall not go into details except to quote part of *Query* 21 in his *Analyst*, where he inquires, "whether the supposed infinite divisibility of finite extension hath not been a snare to mathematicians and a thorn in their sides?" Before this Berkeley had discussed this vital question quite fully in his *Principles of Human Knowledge*, first printed in 1710, then again in 1734. Infinite divisibility, because inapprehensible by our senses, is dismissed from his philosophy as void of meaning or involving contradictions. We quote the following:[2]

"And, as this notion is the source from whence do spring all those amusing geometrical paradoxes which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning. . . . Of late speculations about Infinites have run so high, and grown to such strange notions, as have occasioned no small scruples and disputes among the geometers of the present age. Some there are of great note who, not content with holding that finite lines may be divided into an infinite number of parts, do yet farther maintain that each of those infinitesimals is itself subdivisible into an infinity of other parts or infinitesimals of a second order, and so on *ad infinitum*. These, I say, assert there are infinitesimals of infinitesimals of infinitesimals, etc., without ever coming to an end; so that according to them an inch does not barely contain an infinite number of parts, but an infinity of an infinity of an infinity *ad infinitum* of parts. Others there be who hold all orders of infinitesimals below the first to be nothing at all; . . . Have we not therefore reason to conclude they are *both* in the wrong, and that there is in effect no such thing as parts infinitely small, or an infinite number of parts contained in any finite quantity? . . . If it be said that several theorems undoubtedly true are discovered by methods in which infinitesimals are made use of, which could never have been if their existence included a contradiction in it—I answer that upon a thorough examination it will not be found that in any instance it is necessary to make use of or conceive infinitesimal parts of finite lines, or even quantities less than the *minimum sensible;*

[1] A. C. Fraser, *The Works of George Berkeley*, D.D., Vol. II, Oxford, 1871, p. 501, also Vol. III, pp. 76, 91.

[2] Berkeley, "Principles of Human Knowledge," *The Works of George Berkeley*, Vol. I, pp. 220, 223–225.

nay, it will be evident this is never done, it being impossible. And, whatever mathematicians may think of fluxions, or the differential calculus and the like, a little reflection will show them that, in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense."

The *Analyst* gave rise to a spirited discussion. An anonymous reply by Philalethes Cantabrigiensis appeared under the title, *Geometry, no Friend to Infidelity; or, A Defence of Sir Isaac Newton and the British Mathematicians*, London, 1734. The authorship of this letter has been attributed to Conyers Middleton and Robert Smith, but George A. Gibson makes it plain that the author is James Jurin,[1] a noted Cambridge physician and an admirer of Newtonian philosophy, which he had imbibed from Newton himself. Philalethes admits that the doctrine of fluxions is involved in difficulties, but claims that they are not insuperable. Gibson calls this reply "an extremely weak defence" of the doctrine of fluxions. In 1735 Berkeley published his *Defence, etc.*, alluded to above, to which Philalethes replied in a pamphlet entitled *The Minute Mathematician: or the Freethinker no just Thinker.* Berkeley did not make answer to this, nor to a publication of the same year by Benjamin Robins, a mathematician and military engineer, which appeared in London under the title: *A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions and of Prime and Ultimate Ratios.* A controversy arose between Philalethes and Robins which bears more closely on our present topic than that between Philalethes and Berkeley. Robins and Philalethes differed in the interpretation of Newton; they "began that long struggle in which," as Gibson puts it, "Robins proved his immense superiority to his antagonist, alike in temper, in general mathematical learning, and in special knowledge of Newton's fluxionary methods." The part of the debate which interests us just now relates to the variable's reaching its limit. On this point Robins's vision was somewhat circumscribed; he held that no variable could possibly reach its limit. This interpretation of Newton is at variance with that usually accepted. For the purposes of debate it was no doubt easier for Robins to limit himself to variables which do not reach their limits; from the standpoint of mathematical theory which should be broad enough to explain all ordinary phenomena of motion, his position was unfortunate. Says Robins:

"It was urged that the quantities or ratios, asserted in this method to be ultimately equal, were frequently such as could never absolutely coincide. As, for instance, the parallelograms inscribed within the curve, in the second *lemma* of the first book of Sir Isaac Newton's *Principia*, cannot by any division be made equal to the curvilinear space they are inscribed in, whereas in that *lemma* it is asserted that they are ultimately equal to that space.

"Here two different methods of explanation have been given. The first, supposing that by ultimate equality a real assignable coincidence is intended, asserts that these parallelograms and the curvilinear space do become actually, perfectly, and absolutely equal to each other."

This last view described by Robins was the view of Jurin. No doubt Jurin followed more nearly in Newton's footsteps than did Robins. Newton declares

[1] G. A. Gibson, Review of Cantor's "Geschichte der Mathematik," Vol. 3, in *Proceedings of the Edinburgh Mathematical Society*, Vol. XVII, 1898–99. Gibson's article gives the most complete account of the *Analyst* controversy with which we are familiar.

that the variable becomes "ultimately equal" to its limit, yet Robins insists that he must have seen they would always remain unequal. Robins's contention was hardly valid; whether a variable reaches its limit or not depends wholly upon the variable. Now a law of variation may be artificially established by the human mind. That law may be such that the variable reaches its limit, or it may be such that the variable does not reach its limit. Apparently Robins, perhaps unconsciously, assumed laws of variation which kept the variable and its limit constantly apart, while the great Newton conceived modes of variability not limited to such conditions. How Robins came to insist that his views were those deduced from Newton's *Principia* is elucidated by him in the following passage, in which he says:

> "[Newton] has given such an interpretation of this method as did no ways require any such coincidence. In his explication of that doctrine of prime and ultimate ratios he defines the ultimate magnitude of any varying quantity to be the limit of that varying quantity which it can approach within any degree of nearness, and yet can never pass. And in like manner the ultimate ratio is the limit of that varying ratio."

The reader may compare this passage from Robins with the passages in Newton's *Principia* which we quoted earlier. That Philalethes failed to do Newton justice is clearly brought out by Gibson. According to Philalethes an ultimate ratio is not the *limit* of a varying ratio, but the last value of a ratio. Berkeley very properly argued that there is no last value of the augments except zero, so that the phrase "the ratio with which they vanish," used by Newton himself, does not represent any mathematical operation, and itself requires explanation. Gibson claims that Newton's terminology of first and last ratios was unfortunate, "as it lent itself too readily to an interpretation in the sense of indivisibles; and it was this interpretation that Berkeley and Philalethes alike proceeded upon. Were that interpretation correct, then Berkeley's contentions would in the main be fully justified."

We may sum up the discussion by saying that Berkeley did not directly inquire whether Achilles caught the tortoise or not; that according to the teachings of Newton and Jurin on limits, Achilles did catch the tortoise, though it is not quite evident how the feat was accomplished; that Robins's theory did not allow Achilles to overtake the tortoise, though Achilles would come tantalizingly near doing so.

It was in 1710 that Bayle's famous dictionary was translated from French into English. We are not able to trace any immediate influence of the article on "Zeno of Elea" upon English thought. In 1713 appeared the *Clavis Universalis* of the English divine, Arthur Collier, an idealist who aimed to prove in his book the non-existence of the external world. As his fifth argument he considers motion. He does not mention Zeno, nor any other philosopher, but deserves to rank among Zeno's boldest and most reckless disciples. A few quotations will suffice.[1]

> "A world, in which it is both possible and impossible that there should be any such thing as motion, is not at all;

[1] A. Collier, *Clavis Universalis*, edited by Ethel Bowman, Chicago, 1909, pp. 80–82.

But this is the case of an external world;
Ergo, there is no such world."

". . . Now in such translation the space or line through which the body moved is supposed to pass, must be actually divided into all its parts. This is supposed in the very idea of motion; but this all is infinite, and this infinite is absurd, and consequently it is equally so, that there should be any motion in an external world."

". . . to affirm that a line by motion or otherwise is divided into infinite parts, is in my opinion to say all the absurdities in the world at once. For, *first*, this supposes a number actually infinite, that is, a number to which no unit can be added, which is a number of which there is no sum total, that is, no number at all; consequently, *secondly*, by this means the shortest motion becomes equal to the longest, since a motion to which nothing can be added must needs be as long as possible. This also, *thirdly*, will make all motions equal in swiftness, it being impossible for the swiftest in any stated time to do more than pass through infinite points, which yet the shortest is supposed to do."

Collier gives no evidence of having looked into the higher mathematics as did Berkeley and Hume. After referring to the angle of contact between a circle and its tangent, which "is infinitely less than any rectilineal angle," Hume concludes "that all the ideas of quantity upon which mathematicians reason, are nothing but particular, and such as are suggested by the senses and imagination and consequently, cannot be infinitely divisible."[1]

Zeno's arguments appear to have been discussed but little in England during the second half of the eighteenth century. Charles Hutton refers to them in the article "Zeno, Eleates" in his *Mathematical and Philosophical Dictionary*, London, 1795. He describes the "Achilles" and remarks that "the fallacy will soon be detected," as the time can easily be computed when Achilles will not only have overtaken, but actually passed the tortoise.

Some English mathematicians kept in the path laid out by Newton, by teaching that a variable reaches its limit. Especially is this true of mathematicians at Cambridge, from Jurin to Whewell and from Whewell to Todhunter. Says Whewell:[2] "A magnitude is said to be *ultimately equal* to its Limit; and the two are said to be *ultimately in a ratio of equality.*"

Hutton, who was professor of mathematics at the Royal Military Academy, Woolwich, says, in his *Dictionary*, under the word "Limit," that the variable "can never go beyond it." A different exposition was given by Augustus De Morgan. In the article "Progressions" in the *Penny Cyclopædia* (London, 1841) he says of Achilles: "Let him go as far as he may, he must always come up to where the tortoise was before he can reach the point; so that it requires *an infinite number of parts of time*, but here the sophism quietly introduces *an infinite time* to catch the tortoise." De Morgan establishes the two convergent series, the one for the time, the other for the distance, passed over by Achilles, but he ignores the crucial question as to the reaching of the limit. In the article "Limit" he says that the variable "must never become equal" to its limit. Consequently De Morgan's exposition of limits, as given in these articles, lacked the generality necessary to explain the "Achilles."

On the Continent there prevailed the same diversity of definitions of a limit.

[1] D. Hume, *Essays Moral, Political, and Literary*, London, 1898, edited by T. H. Green and T. H. Grose, Vol. II, p. 129.

[2] William Whewell, *Doctrine of Limits*, Cambridge, 1838, p. 18. See also p. 23.

D'Alembert in 1754 puts no restriction upon the variable reaching its limit;[1] only, the variable must not "surpasser la grandeur dont elle approche." It is well known that there was a time when Lagrange was greatly troubled by the lack of rigor in the foundations of the calculus. He said:

"That method [of limits] has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratio of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become, the one and the other, nothing at the same time."[2]

In the nineteenth century Carnot[3] and Cauchy[4] put no restriction upon variables reaching their limits. In 1817 Bolzano, whose writings did not at the time receive the attention they deserved, was concerned with the limits of continuous functions which attain their limits.[5] Later some French writers thought it necessary to impose restrictions. With Duhamel[6] the variable "never reaches" its limit. In Germany Klügel[7] gives a definition placing no restriction, but in the comments which follow the variable is pictured as not reaching its limit. In 1871 Hermann Hankel starts out in his article "Grenze"[8] by defining what is called a limit in mathematics; the limit is not reached. The difference between the variable and its limit he calls an infinitely small quantity—a quantity no multiple of which is capable of producing unity. But magnitudes of the same kind, by Euclid V, Def. 4, are such that some multiple of one will exceed the other. Hence an infinitely small line is not of the same kind as a finite one. This contradiction is to Hankel one of the indications that a scientific treatment of limits is still wanting. Hankel proceeds to express his adherence to the actual infinite and to develop a more satisfactory definition, free from restriction as to the attainment of the limiting value.

In the United States, as elsewhere, there has been great diversity of practice. Charles Davies of West Point, later of Columbia College, lets the variable reach its limit, in his *Calculus* of 1836. A discussion of this subject was carried on in the *Analyst* by Levi W. Meech, C. H. Judson, De Volson Wood and Simon Newcomb.[9] Wood's article voices the view that prohibiting the variable from attaining its limit "unnecessarily restricts the law of approach of the variable," though the variable can be "subjected to such a law that to the human mind it will appear impossible for it to reach the limit." An elaborate discussion of the

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[1] Article "Limite" in the *Encyclopédie, ou Dictionnaire raisonné des sciences* (Diderot).

[2] Quoted by Bledsoe, and by Carnot in his *Reflexions sur la métaphysique du calcul infinitesimal*, 5. éd., Paris, 1881, p. 147.

[3] Carnot, *op. cit.*, p. 168.

[4] A. L. Cauchy, *Cours d'analyse*, 1821, p. 4.

[5] Philip E. B. Jourdain, "The Development of the Theory of Transfinite Number," *Archiv der Mathematik u. Physik*, Bd. 14, 1909, p. 297. Jourdain's work appears in Bd. 10, 1906, pp. 254–281; Bd. 14, 1909, pp. 289–316; Bd. 16, 1910, pp. 21–43; Bd. 22, 1913.

[6] *Eléments de calcul infinitesimal*, Duhamel, Vol. I, Book. I, Chap. 1.

[7] G. S. Klügel, Mathematisches Wörterbuch, Leipzig, 1805, Vol. II, Art. "Grenze."

[8] *Allgem. Encyklopädie der Wissensch. u. Künste* (Brockhaus), 90. Theil.

[9] *The Analyst* (J. E. Hendricks, Des Moines, Iowa), Vol. I, 1874, p. 133 et seq.; Vol. VIII, 1881, p. 105 et seq.; Vol. IX, 1882, p. 79 et seq.; Vol. IX, 1882, p. 114 et seq.

subject is found in A. T. Bledsoe's *Philosophy of Mathematics*, Philadelphia, 1886. He holds (p. 44) that the variable never actually attains its limit, and

". . . this, I apprehend, will be found to be the case in relation to every variable really used in the infinitesimal method. It will, at least, be time enough to depart from the definition of Duhamel when variables are produced from the calculus which are seen to reach their limit without violating the law of their increase or decrease."

That a teacher who had pondered so long upon the foundations of the calculus as Bledsoe had done, could not think of examples of variables reaching their limits is an indication that the application of the calculus to physics and mechanics did not then receive the careful attention it deserved.

It is with the theory of limits as with negative numbers and imaginaries. In the eighteenth century it was felt that, whether such numbers could exist in algebra, was a matter of argument and demonstration; now it is merely a question of assumption. The same is true with variables reaching their limits. In modern theory it is not particularly a question of argument, but rather of assumption. The variable reaches its limit if we will that it shall; it does not reach its limit, if we will that it shall not. Our "willing" the one thing or the other consists in assuming a continuum in which the limit is a value the variable can assume; our "not willing" consists in not assuming, in the aggregate of values the variable can take, the value of the limit.

A GENERAL FORMULA FOR THE VALUATION OF BONDS.

By C. H. FORSYTH, University of Michigan.

It is the purpose of the present paper to generalize a formula for the valuation of bonds so that it will be applicable to a large number of bond offerings not satisfactorily covered by a known formula.

The problem to be considered may be put more concretely as follows. All formulas known up to the present time, cover the offering of bonds or loans only where the principal is repaid in *equal* installments. The case where payment of the principal is made in a lump sum is included, as a special case.

We shall derive a formula for computing the price of bonds where this principal is repaid in general in *unequal* installments. This formula will include cases where there may be only one installment of any one value. In fact, the formula will include as special cases not only the most general formula known at present but also all the special cases of the latter.

The most general formula[1] for valuation of bonds known up to this time is

$$k = \left(1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{1}$$

where k represents the premium or discount—as the case may be.

The nature of the above bond offering or loan is as follows. The principal

[1] J. W. Glover, *A general formula for evaluating securities*, this MONTHLY, March, 1915.

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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

VII.

7. KANT AND OTHER PRE-CANTORIAN DISCUSSION.

We now come to a commanding figure in philosophic thought—Emmanuel Kant. He took Zeno's dialectics more seriously than had been the custom before. Kant says that critics charged Zeno with a complete denial of both of two self-contradictory propositions. "But," says Kant, "I do not think that he can be rightly charged with this."[1] Zeno was not as much of a skeptic as has been pretended. Kant did not write on Zeno's arguments on motion, but he touched on other arguments of Zeno. Kant's first antinomy, or "the first conflict of the transcendental ideas," contains parts which remind one of the following annihilation of the notion of space, as given by Zeno: If there is space, it is in something, for every thing that is, is in something; but that which is in something, is also in space. Space, then, must also be in space, and so on infinitely: therefore there is no space. While Kant did not contribute directly to a clearer understanding of Zeno's arguments on motion, the effect of his writings was a more painstaking and searching examination of that subject.

In 1794 there appeared in Halle a monograph on Zeno's arguments on motion by C. H. E. Lohse, which is permeated by the atmosphere of Kantian philosophy. It is the earliest publication on our topic which appeared in the form of a monograph.[2] Of its four parts, the first deals with Zeno's system in general, the second gives his arguments against motion, the third elucidates Aristotle's refutation of Zeno, the fourth deals with "the only way" of refuting Zeno. The last argument, the "stade," is not discussed at all. Aristotle's distinction between a potential

[1] *Kant's Werke*, Bd. III, "Kritik der reinen Vernunft," 2. Aufl. (1787), Berlin, 1904, p. 345.

[2] Car. Henr. Erdm. Lohse, *Diss.* (*praeside Hoffbauer*) *de argumentis, quibus Zeno Eleates nullum esse motum demonstravit et de unica horum refutandorum ratione.* Halle, 1794. All our information on Lohse's paper is taken from E. Wellmann, *op. cit.*, pp. 12–14.

and an actual division to infinity is pronounced arbitrary. Whatever can be divided to infinity, says Lohse, actually consists of an infinite number of parts which exist even before division. He decides on this point against Aristotle and in favor of Zeno, as Bayle had done, though he does not mention Bayle. Lohse claims that Zeno's fundamental error lay in a wrong conception of time and space. These are not qualities subject to our senses, but are forms which determine the manner in which our senses are affected; they are *a priori* ideas. Time and space can both be divided to infinity, but one cannot consider time as made up of indivisible points in the manner of Zeno, else what happens in a moment of time would happen in no time. Rest is not, as Zeno and his followers claim, the absence of motion; it is the least velocity of succession. A body can be perceived only as it moves. "Without doubt," says Lohse, "all mistakes of their system sprang from that error. Thence it came that reason and the senses seemed to contradict each other."

Presiding at the time when Lohse presented his dissertation for an academic degree at Halle was Joh. Christoph Hoffbauer (1766–1827) who, many years later, prepared a cyclopædia article, "Achilles (Der Trugschluss)."[1] After expressing his disapproval of Facciolati's argument (previously referred to) he states that Zeno's argument is true only on a condition which has not been stated explicitly: Zeno's contention that the faster runner will always only arrive at the places where the slower has been, and will be behind the slower runner, is true only on condition that the faster has not overtaken the slower. The only thing proved by Zeno is therefore that the faster runner cannot have overtaken the slower as long as the slower is still in advance!

A reply to Hoffbauer's argument was made by Christian Ludwig Gerling, professor of mathematics, astronomy, and physics at the University of Marburg, in a prorectorat address.[2] The claim that Zeno's argument is valid only for certain points, not for all, is no objection at all, unless it is first shown to be a mistake to assert as true for all points what is in fact true of an infinite number of points; a defender of Zeno may always demand that the points be shown, for which the proof does not hold. Gerling insists that Hoffbauer himself reasons in a circle when he accuses Zeno of reasoning in a circle, for whoever has still to prove the possibility of an overtaking is not yet permitted to speak of the time before or after which the overtaking takes place.

Against Waldin's argument, advanced at this same university (Marburg) forty-three years previous, to the effect that Zeno assumes the existence of motion, the very thing that is in dispute, Gerling argues that Zeno's argument is an indirect one, a *reductio ad absurdum*, the form of which is quite valid.

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[1] *Allg. Encycl. d. Wissensch. u. Künste*, von J. S. Ersch u. J. G. Gruber, Leipzig, 1818.

[2] *De Zenonis Eleatici paralogismis motum spectantibus, Dissertatio auctore Chr. Lud. Gerling.* Marburg, 1825. We know this dissertation only from the description of it given by E. Wellmann, *op. cit.*, pp. 14, 15, and by Dr. Johann Heinrich Loewe, "Ueber die Zenonischen Einwürfe gegen die Bewegung," in *Böhm. Gesellsch. d. Wissensch.*, VI Folge, 1 Bd., 1867, pp. 30, 34. In Poggendorff's *Handwörterbuch*, the date of Gerling's dissertation is given as 1830. We have seen references to an edition in German of the year 1846. From this we infer that several editions of it have appeared, and that it enjoyed a considerable circulation.

Lohse's metaphysical apparatus Gerling declares needless and useless. In the constructive part of his dissertation, Gerling dwells on the distinction between continuous and discrete quantity, admits the infinite divisibility of space and time, and constructs the infinite geometric progressions whose sums give respectively the distance and the time of running, before Achilles overtakes the tortoise. Gerling here repeats what Gregory St. Vincent had done long before, only Gerling uses letters, while Gregory assumed a special numerical case. Gregory is nowhere mentioned by Gerling. The sums of the two geometric progressions are values which in no way conflict with the estimate obtained from sensuous perception; Zeno's paradox, as interpreted by aid of the mathematical formulas, conflicts in no way with experience. Hence the puzzle is solved. Though a mathematician, Gerling does not feel the need of explaining the possibility of a variable reaching its limit.

As to the "Arrow" a sharp distinction between the continuous and the discrete is sufficient. In continuous quantity the number of possible subdivisions is *arbitrary*, and each subdivision is itself continuous. Hence Zeno's alleged denial of the infinite divisibility does not follow. Gerling treats the "Stade" with more than customary respect, and admits that, if one assumes with Zeno that space and time be not infinitely divisible, then it follows, as Zeno says, that half the time is equal to the whole time.

An entirely different type of discussion, more along the lines of Kant, profound and obscure, is given by Georg Wilhelm Friedrich Hegel. He holds the view that "Zeno's dialectic of matter has not been refuted to the present day; even now we have not got beyond it, and the matter is left in uncertainty."[1] He protects Aristotle against Bayle who objected to Aristotle's distinction between a potential and an actual subdivision of a line to infinity. Hegel keenly realizes the speculative importance of Zeno's paradoxes and points out that the dialectician of Elea had analyzed our concepts of time and space and had pointed out the contradictions involved therein; "Kant's antinomies do no more than Zeno did here."[2] Movement appears "in its distinction of pure self-identity and pure negativity, the point as distinguished from continuity."[3] This continuity is an absolute hanging together, an annihilation of all differences, of being by itself; the point on the other hand is pure existence by itself, the absolute distinctness from others, the suspension of all self-identity and all hanging together. In time and space the opposites are united in one, hence the contradiction as exhibited in motion. Hegel's position is a long way, still, from Georg Cantor's continuum, with its skilful union of continuity and discreteness. In the "dichotomy" the assumption of half a space is incorrect, says Hegel, "there is no half of space, for space is continuous; a piece of wood may be broken into two halves, but not space, and space only exists in movement."[4] Motion is connectivity, disintegration into an indefinite number of aggregates is its opposite.

[1] G. W. F. Hegel, *History of Philosophy*, transl. by E. S. Haldane, Vol. I, London, 1892, p. 265. See also Hegel's *Sämtl. Werke*, Bd. 13, 1833, pp. 314–327.

[2] Hegel [ed. Haldane], Vol. I, p. 277.

[3] Hegel, *op. cit.*, Vol. I, p. 268.

[4] Hegel, *op. cit.*, Vol. I, p. 271.

Somewhat more specific and comprehensible are the ideas set forth by his philosophical opponent, Johann Friedrich Herbart. Zeno's paradoxes are taken up by him in two works, his popular *Einleitung in die Philosophie* (1813) and his more technical and scientific *Allgemeine Metaphysik* (1828–9). Only in the latter work is the solution of the contradictions attempted. From it we quote:[1]

"The argument inevitably confuses those, who admit the infinite divisibility of the path and then console themselves with a corresponding infinite divisibility of the time, to such a degree that though at first willing to consider the process of dividing, which must continue to infinity, they soon in one leap consider the infinite number of time intervals as passed over, since they see that they must combine the infinite number of parts of the time as well as of the path to the place of overtaking, which they cannot do. The leap and the doubly infinite division are both faulty and amount to naught."

Thus, this infinite subdivision of the time and space is rejected by Herbart, because the mind is not able to *imagine* all the steps in the process. Imaginability is made the criterion of truth or error. This criterion throws out infinity at once; it throws out non-euclidean geometry and other parts of mathematics. We cannot really imagine things which we have never seen. Our senses are inaccurate, our intuitions are crude; hence it would seem to us impossible to build up sound mathematical theory, if everything unimaginable were to be cast aside. Herbart tries to explain motion by the concept of *velocity,* which seems itself to involve a contradiction, that Herbart endeavors to resolve by his theory of a "rigid line," a sort of continuum, which might have given rise to great possibilities upon more careful development. As it is, it offers greater obstacles by far than does the original "Dichotomy" or "Achilles" which it is intended to explain.

A still different attitude toward Zeno's paradoxes is taken by Friedrich Adolf Trendelenburg of the University of Berlin, in his *Logische Untersuchungen,* 1840, where he constructs his philosophic system upon the concept of motion. Constructive motion is common to the external world of being and to the internal world of thought, so that thought, as the counterpart of external motion, produces from itself space, time, and the categories. Motion is undefinable. In accordance with this view it is only through motion that Zeno's arguments against motion have come to be. For they depend upon the division of time and space, and the synthesis of those divisions. But division and synthesis are nothing but special forms of motion. What the proofs combat, they themselves use as the means of combat, and thereby testify to the controlling nature of motion. Trendelenburg and Kant evidently begin at opposite ends; Kant takes time and space as *a priori* ideas, and motion as secondary and dependent upon them; Trendelenburg makes motion the *a priori* idea, and pretends to derive time and space from it.

Friederich Ueberweg of the University of Königsberg[2] refers to our subject in different parts of his *Logik.* He says in one place that the "Achilles" proves too little; it proves merely that the tortoise cannot be overtaken within a definite series, and then claims that the tortoise cannot be overtaken anywhere and at

[1] J. T. Herbart, *Sämtl. Werke,* herausgeg. von Karl Kehrbach, Langensalza, Vol. VIII, 1893, p. 177.

[2] F. Ueberweg, *System der Logik,* 2. Aufl., p. 387 ff.

any time. True as this criticism may be, it does not illuminate the matter sufficiently to satisfy the reader.

Of the same type, but fuller in statement, is a criticism by Carl Prantl,[1] professor at the University of Munich. He claims that Zeno discarded the concept of continuity by considering only some particular points on a line and only some particular moments in time. By drawing his inferences from these disintegrated fragments of time and space, Zeno was able to advance contradictions in a picturesque manner. This conversion of the general and continuous into the particular and momentary will be encountered often, says Prantl, in those who care more for rhetorical form than for true philosophy.

Much confidence in his ability to clear up the mystery of Zeno's paradoxes is displayed by Eugen Carl Dühring in his *Kritische Geschichte der Philosophie*, first edition, 1869. Three concepts are necessary here: rest, motion and position. Usually only the first two are considered. At each moment (point) of time a moving body has a definite position but no motion. This fact makes it difficult to explain motion. He says further:[2] "The compelling force and real conclusiveness of the Eleatic contentions is to be found chiefly and almost exclusively in the logical necessity which does not permit the infinite to be thought of as completed, as enumerated so to speak, and concluded. . . . It is the concept of infinity which proves itself everywhere and also where it is not readily recognized, as the true cause of the contradictions." Dühring discusses infinity in several places of his works. He believes in the infinity usually set forth in the study of the calculus,—a variable which increases without limit, but at any moment has really a finite value. He makes war against the concept of an actual infinity—"jene wüste, sich widersprechende Unendlichkeit." "The infinite divisibility indicates . . . only this, that I can conceive the division of a quantity as far as I choose, without limit. If, on the other hand, I consider the division to infinity as really existing outside of my presentation of it, then there soon result the most manifold contradictions. . . . As regards motion, it must be recognized that it belongs to the empirical concepts, i. e., in our thinking there remains here always an unrecognizable residue, for we must give up the attempt to penetrate to the reasons of the phenomena." Georg Cantor criticizes Dühring in these words:

> "The proofs of Dühring against the properly-infinite could be given in much fewer words and appear to me to amount to this, either that a definite finite number, however large it may be thought to be, can never be an infinite number, as follows immediately from the concept of it, or else that the variable, an unlimitedly large finite number, cannot be thought of with the quality of definiteness and therefore not with the quality of existence, as follows again from the nature of the variability. That not the least is hereby established against the conceivability of transfinite number, I feel certain; and yet, those proofs are taken as proofs against the reality of transfinite numbers. To me this mode of argumentation appears the same as if, from the existence of innumerable shades of green, we were to conclude that there can be no red."[3]

Dühring's explanation of infinity and of Zeno is accepted by Eduard Wellmann, in his historical monograph[4] of 1870. Another research, partly historical

[1] Carl Prantl, *Geschichte der Logik im Abendlande*, 1. Bd., Leipzig, 1855, pp. 10, 11.

[2] *Kritische Gesch. d. Philosophie*, Dr. E. Dühring, Leipzig, 1894, p. 49.

[3] Georg Cantor, *Grundlagen einer allg. Mannichfaltigkeitslehre*, Leipzig, 1883, p. 44.

[4] E. Wellmann, *op. cit.*, p. 23.

and partly expository, was published in 1867, by Johann Heinrich Loewe, a pupil of the philosopher, Anton Günther of Vienna. It is referred to by Knauer[1] as the most acute and satisfactory explanation that has yet been offered. "The solution of the riddle," says Loewe,[2] "appears to us to lie in the knowledge that contradictions must arise inevitably, as soon as space, time, and motion are considered at the same time from the stand-point of sensuous presentation and of non-sensuous conceptual reasoning." One point of view appeals to the imagination; the other to abstract thought. Sensuous perception can follow the process of infinite division only a little way, everything beyond is a matter of pure reason. Gerling's presentation of "Achilles" is an appeal to reason. As long as one considers the infinite multiplicity of small distances and of time-intervals, one approaches the riddles from the standpoint of abstract thought; when one appeals to the imagination, then the finite time and the finite length of the race stand out. Loewe seems still to hold to the old view that thought can recognize no end to a motion which extends over an infinite process. Hence the contradiction must stand, the antinomy is evident.

Thus we see that German philosophy down to the last quarter of the nineteenth century continually accentuates the existence of contradictions in the problem of motion.

Some English thinkers of the nineteenth century, who were interested in Zeno's arguments, came under the influence of Kantian philosophy. The Kantian attitude toward Zeno is described in the article "Zeno" in the eighth edition of the *Encyclopaedia Britannica* (1860) thus:

"He brought a most powerful mind to his task, and, curious to say, subsequent thinkers have very generally agreed in misunderstanding both his reasoning and his method, and it is only of late years that Kant, in his *Antinomies of the Pure Reason* (see *Kritik der Reinen Vernunft*) seized upon the much maligned doctrines of the Eleatic, and held them up to the admiration of all true thinkers as rare examples of acute and just thought. Bayle, in a clever paper on Zeno, in his *Dictionnaire*, makes him, according to custom, a sceptic. Brucker finds that Zeno surpasses his intelligence, and he is content to make him a pantheist. Others again, have charged him with nihilism. Zeno, fortunately, can afford to sit quite easy to all those affronts offered to his reason . . . they [arguments against motion] all take their rise, as Kant and Hamilton (*Lectures on Metaphysics*) have shown, from the inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility, of space and time, as extensive and as protensive quantities. The possibility of motion, however certain as an observed fact, is thus shown to be inconceivable. To have discovered this peculiarity of our mental constitution, and to have stated it with eminent clearness, belongs to Zeno the Eleatic, and to him alone."

Sir William Hamilton puts this matter thus:[3]

"Time is a protensive quantity, and, consequently, any part of it, however small, cannot, without a contradiction, be imagined as not divisible into parts, and these parts into others *ad infinitum*. But the opposite alternative is equally impossible; we cannot think this infinite division. One is necessarily true; but neither can be conceived possible. It is on this inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility of space and time, that the arguments of the Eleatic Zeno against the possibility of motion are founded,—arguments which at least show, that motion, however certain as a fact, cannot be conceived pos-

[1] Vincenz Knauer, *Die Hauptprobleme der Philosophie*, Wien u. Leipzig, 1892, p. 54.

[2] J. H. Loewe, *op. cit.*, p. 32.

[3] *Lectures on Metaphysics and Logic*, by Sir William Hamilton, Vol. I, Boston, 1863, Lecture 38, p. 530.

sible, as it involves a contradiction. . . . Now the law of mind, that the conceivable is in every relation bounded by the inconceivable, I call the Law of the Conditioned."

John Stuart Mill, in his *Logic*,[1] refers to Thomas Brown who considered the "Achilles" insoluble, and then offers a solution to the invention of which he lays no claim. It presents no new points of view. Herbert Spencer discusses questions of time and space in his *First Principles*[2] and concludes in general that "ultimate scientific ideas, then, are all representative of realities that cannot be comprehended." In particular, "halve and again halve the rate of movement for ever, yet movement still exists; and the smallest movement is separated by an impassable gap from no movement."

It is readily seen that the nineteenth century philosophers had penetrated deeper than most of their predecessors and had encountered difficulties previously neglected by Hobbes and others who seemed to think that they had solved the "Achilles" paradox by the mere statement that time, as well as space, was infinitely divisible. What came to be thoroughly realized since the time of Kant was the impossibility of *imagining* the "Achilles" from the standpoint of infinite divisibility of a distance, that all appeals to intuition were futile. When Spencer says that infinite divisibility cannot be "comprehended," and Thomas Brown and Sir William Hamilton say that motion is "insoluble" and "inconceivable," I take it that they mean simply that these processes are *unimaginable*, that they are beyond the reach of our sensual intuitions. I do not interpret them to mean that these processes are beyond the reach of logic, beyond the reach of the reasoning faculty so as to be, and forever remain, wholly mysterious. Mathematics includes among its results numerous teachings which one cannot "imagine." Probably no one claims to be truly able to visualize to himself the non-euclidean geometries; analysts do not claim to be able to imagine or see a continuous curve which has no tangent line at any of its points. Yet no modern mathematician rejects non-euclidean geometries and non-differentiable continuous curves.

These unimaginable mathematical creations are admitted into the science as a matter of necessity. Felix Klein states the issue as follows: "As the subjects of abstract geometry cannot be sharply comprehended through space intuition, one cannot rest a rigorous proof in abstract geometry upon mere intuition, but must go back to a logical deduction from axioms assumed to be exact."[3]

It so happens that England's two famous opium eaters, Thomas De Quincey and Samuel Taylor Coleridge, were interested in the "Achilles." Coleridge's critical powers were set forth by De Quincey in the following terms:[4]

"I had remarked to him that the sophism, as it is usually called, but the difficulty, as it should be called, of Achilles and the Tortoise, which had puzzled all the sages of Greece, was, in fact, merely another form of the perplexity which besets decimal fractions; that, for example, if

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[1] *A System of Logic*, Vol. II, London, 1851, p. 381.

[2] H. Spencer, *First Principles of a New System of Philosophy*, New York, 1882, pp. 47–67.

[3] F. Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie*. Leipzig, 1907, p. 19.

[4] *Tait's Magazine*, Sept. 1834, p. 514.

you throw $\frac{2}{3}$ into a decimal form, it will never terminate, but be .666666, etc., *ad infinitum*. 'Yes,' Coleridge replied, 'the apparent absurdity in the Grecian problem arises thus,—because it assumes the infinite divisibility of space, but drops out of view the corresponding infinity of time.' There was a flash of lightning, which illuminated a darkness that had existed for twenty-three centuries."

As a matter of fact, Aristotle had seen that far. But Coleridge proceeded somewhat farther in an essay on Greek sophists in *The Friend*,[1] where he says:

"The few remains of Zeno the Eleatic, his paradoxes against the reality of motion, are mere identical propositions spun out into a sort of whimsical conundrums, as in the celebrated paradox entitled Achilles and the Tortoise, the whole plausibility of which rests on the trick of assuming a *minimum* of time while no *minimum* is allowed to space, joined with that of exacting from *intelligibilia*, *νούμενα*, the conditions peculiar to objects of the senses *φαινόμενα* or *αἰνϑανόμενα*."

What belongs to Coleridge himself in this passage is the contention that the sophism consists in applying to an idea conditions only properly applicable to sensuous *phaenomena*. Coleridge's argument was elaborated many years later in dialogue form, by Shadworth H. Hodgson. We give the critical part of the discussion:[2]

". . . being infinitely *divisible* is not the same thing as being infinitely *divided*. Actually to divide to infinity that hundredth part of a minute, in which (*phenomenally* as you say) Achilles overtakes the tortoise, is an infinitely long operation. . . . And *this* division you call upon Achilles to perform, before the tortoise can be overtaken, and to perform *phenomenally*. . . . You require that Achilles shall exhibit to the senses the infinite divisibility of time and space, which appertains to them truly indeed, but only as objects of imagination and thought. . . . The world of *thought* and *reality* is not a world apart, but is identical with the phenomenal world, only differently treated. . . . Neither is there any contradiction between them. Phenomenal motion is as infinitely divisible in *thought* as time and space are."

This explanation does not explain. Even as "objects of the imagination" the infinite divisibility of time and space is a source of perplexity. Our imagination is unable to follow Achilles to the end, through the infinities of time and space intervals. Moreover, "*thought* and *reality*" are indeed worlds "apart" whenever the time intervals, corresponding to the space-intervals passed over by Achilles, are so taken that they form together an infinite series that is *divergent*, so that, in thought, Achilles never overtakes the tortoise; in Zeno's traditoinal argument, "thought and reality" were "apart."

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[1] *Complete Works of S. T. Coleridge*, Vol. II, New York, 1856, p. 399.
[2] *Mind*, London, Vol. V, 1880, pp. 386–388.

[The remaining parts of this series are: *D*. VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM (G. Cantor); *E*. POST-CANTORIAN DISSENSION.]

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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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VIII.

D. VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM.

The full explanation of Zeno's paradoxes requires two ideas which are very familiar to the modern mathematician, namely, the acceptance of the existence of actually infinite aggregates and the idea of a connected and perfect continuum. The first concept, that of actual infinity, as opposed to potential infinity, had been under contemplation ever since the time of Aristotle. Men like St. Augustine, Galileo, Pascal, Volder, Schultze seemed to have had a more or less non-contradictory conception of it. Others only denied it. Among the latter were both philosophers and mathematicians; the list includes men like Thomas Aquinas, Gerdil, Descartes, Spinoza, Leibniz, Lock, Lotze, Renouvier, Moigno, Cauchy, Gauss, and of course many others.[1] Wonderful insight into this matter was possessed by Galileo. He showed for instance that there were as many integers that were perfect squares as there were integers altogether. Strange to say, Galileo's argument has been misinterpreted by some recent writers. Instead of accepting the conclusion as legitimate for that sort of infinity, as it was accepted by Galileo, these writers declared the conclusion absurd, hence the hypothesis of actual infinity a myth. Among men taking this view was the French philosopher F. Pillon.[2] In 1831 (July 12) the great K. F. Gauss of Göttingen wrote a letter to Schumacher in which he declares himself as opposed to the actual infinity in mathematics.

[1] For details see Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuele Unendliche" in *Zeitschr. f. Philos. u. philos. Kritik*, Bd. 88, p. 224.

[2] *L'année philosophique*, I, 1890, p. 84, quoting among others Cauchy's *Sept leçons de physique générale*, 1868, as follows: "Cette proposition fondamentale, démontrée par Galilée (qu'on ne saurait admettre une suite ou série actuellement composée d'un nombre infini de termes), s'applique aussi bien à une série de termes ou d'objects, qui ont existé." Cauchy was strongly influenced on this matter by the writings of Gerdil.

In the nineteenth century voices in favor of the actual infinity began to speak with greater emphasis. In 1823 John Bolyai, of non-euclidean geometry fame, wrote down that quality of an infinite aggregate: "An infinite aggregate is one equivalent to a part of itself."[1] Another pioneer in this field was the Bohemian mathematician, Bernard Bolzano, whose writings have only in recent years begun to receive proper appreciation.[2] After the appearance of Cantor's writings his ideas received wide recognition among mathematicians, notwithstanding certain perplexing paradoxes to which some of the more advanced developments of the subject gave rise. Among philosophers the Cantorian ideas found slower recognition. The question at issue is usually not so much one of logic, as it is of the postulates which the reasoner is willing to accept as reasonable and useful. An investigator who vetoes any assumption which does not appeal to his intuition or to his power of imagination can hardly find comfort in Cantor's theory of aggregates and the Cantor continuum. To him Zeno's paradoxes must necessarily remain paradoxes forever.

The second notion needed for the full elucidation of our subject is the "connected" and "perfect" continuum, which we owe to Georg Cantor and Dedekind. To their names should be added that of Karl Weierstrass who banished from analysis the mystical notion of the infinitesimal as a constant smaller than any assignable number, defying the Archimedian postulate. We are not aware that any of these three men wrote directly on the paradoxes of Zeno. But they laid the foundation on which a rational theory of them rests. Richard Dedekind brought out two wellknown publications: *Stetigkeit und irrationale Zahlen*, Braunschweig, 1872, and *Was sind und was sollen die Zahlen*, Braunschweig, 1888. Georg Cantor's first important publication on the theory of aggregates is his *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, Leipzig, 1883. The fundamental ideas advanced by Dedekind and Georg Cantor are so easily accessible and so generally known, that no account of them is needed here. At first British mathematicians took little interest in Cantor's developments. Only in recent years have they been taken up in Great Britain. An unusually interesting outline of them is given by Ernest William Hobson in his presidential address "On the Infinite and the Infinitesimal in Mathematical Analysis," before the London Mathematical Society, in 1902.[3] We quote the following:

"When it is conceived that these mere potentialities pass into actualities, that *fixed* numbers or magnitudes exist which are infinite or infinitesimal, that the mere indefinitely great becomes an actual infinite, or the merely indefinitely small becomes an actual infinitesimal, the region of serious controversy has been reached. . . .

"Here we have the origin of the method of limits, in its geometrical and its arithmetical forms, and here we come across the central difficulty of the mode in which a limit was regarded as being actually attained. A limit which appeared only as the unattainable end of a process of indefinite regression, and to which unending approach was made, had, by some process inaccessible to the sensuous imagination, to be regarded as actually reached; the chasm which separated the limit from the approaching magnitudes had in some mysterious way to be leapt over. . . .

[1] See Halsted's *Bolyai's Science Absolute of Space*, 4th Ed., 1896, § 24, p. 20.

[2] See H. Bergmann, *Das philosophische Werk Bernard Bolzanos*, Halle, 1909; also F. Přihonský, *Dr. Bernard Bolzano's Paradoxien des Unendlichen*, Berlin, 1889.

[3] *Proceedings London Mathematical Society*, Vol. 35, London, 1903, p. 117.

"The notion of number, integral or fractional, has been placed upon a basis entirely independent of measurable magnitude, and pure analysis is regarded as a scheme which deals with number only, and has, *per se*, no concern with measurable quantity. Analysis thus placed upon an arithmetical basis is *characterized by the rejection of all appeals to our special intuitions of space, time and motion, in support of the possibility of its operations.* . . ."

"By this conception of the domain of number the root difficulty of the older analysis as to the existence of a limit is turned, each number of the continuum being really defined in such a way that it itself exhibits the limit of certain classes of convergent sequences. . . . It should be observed that the criterion for the convergence of an aggregate is of such a character that no use is made in it of infinitesimals, definite finite numbers alone being used in the test. . . .

"This [old] intuitive notion of the continuum appears to have as its content the notion of unlimited divisibility, the facts that, for instance, in the linear continuum we can within any interval PQ find a smaller one, $P'Q'$, that this process may be continued as far as the limits of our perception allow, and that we are unable to conceive that even beyond the limits of our perception the process of divisibility in thought can come to an end. However, the modern discussions as to the nature of the arithmetic continuum have made it clear that this property of unlimited divisibility, or connexity, is only one of the distinguishing characteristics of the continuum, and is insufficient to mark it off from other domains which have the like property. The aggregate of rational numbers, or of points on a straight line corresponding to such numbers, possess this property of connexity in common with the continuum, and yet it is not continuous." . . .

"The other property of the aggregate which is characteristic of the continuum, is that of being, in the technical language of the theory of aggregates (Mengenlehre) perfect: the meaning of this is that all the limits of the converging sequences of numbers or parts belonging to the aggregate themselves belong to the aggregate; and, conversely, that every number or point of the aggregate can be exhibited as the limit of such a sequence. . . .

". . . the latter property of the continuum, which was not brought to light by those who took the intuitive continuum as a sufficient basis, is in some respects the more absolutely essential property for the domain of a function which is to be submitted to the operation of the calculus."

"In order to exhibit the way in which transfinite ordinal numbers are required when we deal with non-finite aggregates, I propose to refer to a well-known paradox of Achilles and the tortoise. . . . Let us indicate the successive positions of Achilles referred to, by the ordinal numbers 1, 2, 3, . . . suffixed to the letter A, so that $A_1\ A_2\ A_3$ . . . represent the positions of Achilles. . . . These points $A_1\ A_2$

$$\begin{array}{lllllllll} B_2 & & B_3 & B_4 & & & & & \\ \hline A_1 & & A_2 & A_3 & A_4 & A_\omega & A_{\omega+1} & A_{\omega+2} & A_{\omega 2} \end{array}$$

A_3 . . . have a limiting point, which represents the place where Achilles actually catches the tortoise. The limiting point is not contained in the sets of points $A_1\ A_2\ A_3$ . . .; if we wish to represent it, we must introduce a new symbol ω, and denote the point by this number. It does not occur in the series 1, 2, 3, . . . but is preceded by all of these numbers, and yet there is no number immediately preceding it; it is the first of a new series of numbers."

Hobson proceeds to show how a finite number can maintain itself against a transfinite ordinal number, by showing that $\omega = 1 + \omega$, but $\omega + 1 > \omega$; the commutative law in addition is seen to fail. He brings out the necessity for the introduction of transfinite numbers for the representation of the limit which is not itself contained within the region of the convergent process. Hobson's exposition represents an explanation which recent developments of mathematics offer of the "Dichotomy" and the "Achilles." No doubt, some readers might have desired a fuller exposition of details. It will be noticed that the time-element was not considered by Hobson at all. The Dedekind and Georg Cantor theories of the continuum do not involve the element of time. But how is it possible to ignore time in questions involving motion? In the first place it is pointed out by Cantor[1] that the continuum is a much more primitive and general

[1] *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, von Georg Cantor, Leipzig, 1883, p. 29.

concept than the concept of time, that the theory of the continuum is needed for a clear exposition of time or of any independent variable, that time cannot be considered as the measure of motion; on the contrary, time is measured by motion—the motions of heavenly bodies, the motions of the hands of a watch or clock, the displacement of sand in the hour glass. In the second place the consideration of time is not needed at the critical point where the ability of Achilles to overtake the tortoise is under consideration. Suppose the tortoise has an initial start of 10 ft. and that it travels 1 ft. per second, while Achilles travels 10 ft. per second. Forming the series A, whose terms represent each the distance Achilles travels to come up to the place where the tortoise was at the beginning of the time interval under consideration, and letting the series T represent these time-intervals, we have

$$A \qquad 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots,$$

$$T \qquad 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots.$$

Both geometric infinite series are convergent; the sum of the terms of each series approaches a finite number as a limit. Now comes the ever-present, delicate question whether the sum actually *reaches* its limit. It is to be observed that this question arises in each series, that one series does not help the other. If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the "Achilles." Whether the sum of A reaches its limit or not is a matter of pure assumption on our part. If the limiting value $11\frac{1}{9}$ ft. is assumed to be included in the aggregate of numbers which the distance-variable may take, then of course the variable reaches its limit; if $11\frac{1}{9}$ ft. is not assumed as a value which the variable may take, then of course the limit is not reached. It is here that we must receive a suggestion from our sensuous observations; we know from our knowledge of motion as supplied to us by our senses that Achilles travelling with a uniform finite velocity in the same direction will within a finite time reach the distance $11\frac{1}{9}$ ft. from his starting point. This information, supplied to us by our senses, enables us to choose, of the two possible alternative assumptions offered by theory (as mentioned above), the one which makes the variable sum A conform with the known sensuous phenomena. On this assumption the region of the convergent process has a limit which is contained in the aggregate of values the variable can take, and, as explained by Hobson, the limit is *reached by the variable.* Viewed from the standpoint of the theory of infinite aggregates and of the Georg Cantor conception of the continuum, the "Achilles" is almost a self-evident proposition. Sensuous knowledge suggests that we make the aggregate of values of the variable distance travelled by Achilles a *perfect* aggregate; then theory tells us that in a perfect aggregate every converging process has a limit which is reached.

Interesting remarks on the Georg Cantor continuum are found also in Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, p. 51:

"The term 'arithmetic continuum' is used to denote the aggregate of real numbers, because it is held that the system of numbers of this aggregate is adequate for the complete analytical representation of what is known as continuous magnitude. The theory of the arithmetic continuum has been criticised on the ground that it is an attempt to find the continuous within the domain of number, whereas number is essentially discrete. Such an objection presupposes the existence of some independent conception of the continuum, with which that of the aggregate of real numbers can be compared. At the time when the theory of the arithmetic continuum was developed the only conception of the continuum which was extant was that of the continuum as given by intuition: but this, as we shall show, is too vague a conception to be fitted for an object of exact mathematical thought, until its character as a pure intuitional datum has been modified by exact definitions and axioms."

It will be seen as we proceed that the objection to the Georg Cantor continuum, to which Hobson refers, is frequently made. It is the general objection that a line which is continuous cannot possibly be constructed out of mathematical points, external to each other. Perhaps this general objection which naturally suggests itself at the very start has discouraged non-mathematicians from going to the trouble of studying the Cantor continuum with the care necessary for its comprehension. Philosophers who have subjected themselves to such study have been amply repaid for their labor. They have found it to be a device of the understanding "whereby we give conceptual unity and an invisible connectedness to certain types of phenomenal facts which come to us in a discrete form and in a confused variety."[1]

The other important shift in the point of view, made by the creators of the modern linear continuum, was the rejection of all infinitesimals, that is of quantities which do not obey the Archimedian postulate. This postulate says that if a and b are two numbers (not zero), such that $a < b$, then it is always possible to find a finite integer n so that $na > b$. The infinitesimal which had been the subject of many controversies and was regarded by many as containing an element of mysticism, was banished by Weierstrass and Cantor from their mathematical concepts. In former years the infinitesimal was considered as necessary in the explanation of the linear continuum. Johann Heinrich Lambert wrote to Holland in a letter of April 7, 1766, on the "angle of contact" as follows:[2]

"Do you believe, my dear Sir, that one can dispense with the concept of the infinitely small in the concept of continuity? . . . Continuity demands that this variation be less than every assignable quantity. It is thus impossible to estimate this variation by a finite quantity, and equal to 0 it can not be either. There seems therefore nothing left than to say that the change in direction is infinitely small."

The impossibilities of one generation often become the possibilities of a succeeding generation. Weierstrass's banishment of the infinitely small has found wide following; the old-time infinitesimal is no longer needed in explaining the continuum. The rejection of the infinitely small is looked upon by such mathe-

[1] H. Poincaré, *The Foundations of Science*, transl. by G. B. Halsted, New York, 1913, introduction by Josiah Royce, p. 16.

[2] *J. H. Lamberts deutscher gelehrter Briefwechsel*, Vol. I, Berlin, 1781, p. 141.

matical philosophers and logicians as Bertrand Russell[1] and A. N. Whitehead[2] as steps toward greater mathematical rigor. It must be emphasized, however, that the school of Weierstrass has not found universal recognition; there are modern champions of the infinitely small, chief among whom is the Italian mathematician Giuseppe Veronese. They insist that the Cantor continuum is not the only possible non-contradictory continuum and proceed to construct a higher and more involved, non-archimedean, continuum in which infinitely small distances are given. This is not the place for attempting a minute statement of the controversy between the two schools; the controversy, by the way, has no national aspect. There have been followers of Veronese in Germany (for instance, Stolz, Max Simon), and followers of Weierstrass and G. Cantor in Italy (for instance, Peano). So far as we have noticed, the Zeno arguments have not been studied and given explicit treatment on the basis of the Veronese continuum.[3] In America C. S. Peirce has adhered to the idea of infinitesimals in the declaration: "The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction."[4] Apparently, before he had acquired familiarity with the writings of Dedekind and Georg Cantor, C. S. Peirce had firmly recognized that for infinite collections the axiom, that the whole is greater than its part, does not hold.

[To be continued]

ON NAPIER'S FUNDAMENTAL THEOREM RELATING TO RIGHT SPHERICAL TRIANGLES.

By ROBERT MORITZ, University of Washington.

In view of the recent celebration of the tercentenary of the publication of Napier's greatest work, the "Mirifici logarithmorum canonis descriptio," it is highly fitting that his rule for the circular parts should be rescued from the rubbish heap of mnemotechnics and be assigned its proper place as the most

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[1] See, for instance, his article in the *International Monthly*, Vol. 4, 1901, p. 84 and seq.

[2] A. N. Whitehead, *Introduction to Mathematics*, New York and London, 1911, pp. 156, 226–229.

[3] References to this controversy are as follows: G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, übersetzt v. A. Schepp, Leipzig, 1894, Anhang, p. 631–701; Max Simon, "Historische Bemerkungen über das Continuum, den Punkt und die Gerade Linie," *Atti del IV. Congresso Internazionale dei matematici*, Roma, 1908, pp. 385–390; G. Cantor's letter to Vivanti, *Rivista di mat.* V, 104–108; G. Cantor's letter to Peano, *Rivista di mat.* V, 108–109; G. Cantor, "Zur Begründung der Transfiniten Mengenlehre I," *Mathematische Annalen*, Vol. 46, 1895, page 500; Frederico Enriques, *Probleme der Wissenschaft*, 2. Teil, übersetzt von K. Grelling, Leipzig und Berlin, 1910, pp. 324–329. An able discussion of infinity, infinitesimals and the continuum is given by Josiah Royce, a philosopher familiar with mathematical thought, in his *The World and the Individual*, New York, 1900, pp. 505–560. See also G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitsch. für Philosophie u. Philosophische Kritik*, Vol. 91, Halle, 1887, p. 113; O. Stolz in *Mathematische Annalen*, Bd. XVIII, p. 699, also in *Berichte des naturw.-medizin. Vereins in Innsbruck*, Jahrgänge 1881–82 und 1884, also in *Vorlesungen über allgem. Arithm.*, Leipzig, 1. Theil, 1885, p. 205.

[4] C. S. Peirce, "The Law of Mind" in *The Monist*, Vol. 2, 1892, p. 537.

It is proposed to send out the notice early in December with the names of all signers received up to that time. The meeting will be called at Columbus, Ohio, in connection with the holiday convocation of the American Association for the Advancement of Science. The name of the new society, its precise character and policy, its relation to the AMERICAN MATHEMATICAL MONTHLY, etc., will be questions for full discussion and determination at the organization meeting.

It should be clearly understood that this whole movement is a matter of public concern, and is in no sense a private undertaking; nor is it an effort on the part of those interested in the MONTHLY to rescue it from impending bankruptcy. The MONTHLY is in sound financial condition and is seeking no rescue measures. Its friends and supporters are interested in this new movement for the same reasons which actuate the rest of the signers to the call for the organization meeting; namely, a sincere desire to promote the course of mathematics in this country in all its many and varied aspects, and especially in that field that has been so greatly neglected,—the field of collegiate mathematics.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

IX.

E. POST-CANTORIAN DISSENSIONS.

Very frank and brilliant in its mode of exposition but perhaps lacking the originality and depth of the writings of Dedekind and Georg Cantor is the *Allgemeine Functionentheorie* of Paul du Bois-Reymond, Tübingen, 1882. This book, which is contemporaneous with Cantor's creation, discusses the philosophy and theory of the fundamental concepts of quantity, limit, argument, function. He declares that the difficulties surrounding the idea of a limit are not mathematical in character but have their roots in the "simplest parts of our thinking, our conceptions or images" (Vorstellungen). He says that there are two conceptions, those of the idealist and those of the empiricist, "which have equal right to count as fundamental views of rigorous science,"[1] for neither yields contradictory results, at least in pure mathematics. The author presents both sides "with equal rigor," and does not award victory to either. The idealist defends the existence not only of what can be imagined, but also of things unimaginable;[2] he assumes a transcendental attitude. Accordingly he assumes the termination of series, such as those given by endless decimal fractions, which are really "given"

[1] Paul du Bois-Reymond, *Die Allg. Functionentheorie*, Tübingen, 1882, p. 2.
[2] P. du Bois-Reymond, *op. cit.*, pp. 110, 111.

only to a certain term; he recognizes the limit of these decimals. The empiricist says on the other hand, "everything that is scientifically established springs from sense-perception; whatever is unimaginable must be rejected; we must proceed from images to images (von Vorstellungen zu Vorstellungen);[1] he denies the existence of a limit of an endless decimal fraction and is satisfied to take into account the members of the decimal as far as you may wish. A continuum such as that of G. Cantor and Dedekind does not exist, because it cannot be imagined. Take the endless decimal $0.\alpha_1\alpha_2\alpha_3\cdots$,

"In a drawing . . . imagine the ends of the distances $0.\alpha_1$, $0.\alpha_1\alpha_2$, $\cdots$ marked off from the zero point and designated by fine, short lines drawn across the line segment. These cross lines become more and more dense, and they must sometime stop, because the means of drawing fail us and, at any rate, one cannot draw an unlimited number of marks. There follows now after a short interval following the mark $0.\alpha_1\alpha_2\cdots\alpha_p$, which we shall call the fog-interval, a fine mark, somewhat longer than the previous ones, which represents the limit. In the fog-interval before the limit all sorts of philosophic apparitions play their pranks. Here play the celebrated sophisms; it is this that the idealist resolves into the *quantitates infinitesimas*. The empiricist's drawing looks a little different. The marks are continued, even if the means of drawing do not permit the marks to remain distinct. Then the marks run into each other, to show that their width is greater than the distance of $0.\alpha_1\alpha_2\cdots\alpha_p$ from the limit. The drawing of further marks is stopped when a further advance of the marks can no longer be recognized and new marks would be drawn through the same place. That is then the limit."[2]

The process of a variable continually approaching but never reaching its limit is characterized by du Bois-Reymond as follows:[3]

"Between the conception of the idealist, who lets dx be infinitely small and connects it with the idea of something at rest, unchanging, and my conception which assumes dx finite and sufficiently small but likewise at rest, there slips in a third conception (mentioned page 83, 84) in which, as is commonly expressed, dx is a quantity in the act of disappearing (quantité évanouissante), hence a quantity which is continually varying toward zero. The idea of a quantity in continual flux is repugnant to me. It goes against my grain to have symbols in my formulas for quantities which set themselves in motion as soon as I look at the formulas, and hasten toward zero, which, however, they are permitted to reach only at the end of the computation. As long as the book is closed, there reigns profound silence. No sooner do I open it than there begins that race toward zero of all quantities affected by a d."

Feelings of this sort have doubtless come to many mathematicians of the older school. This perpetual approach without ever reaching the goal is wearisome. Why not divorce the variable and its limit from the limitations of time? Or if, as in mechanical problems, the idea of time seems difficult to eliminate, hasten the successive steps of approach to the limit, the times for these steps diminishing at a sufficiently rapid rate, so that the time elements form together a converging series. Thereby the limit is reached in a finite time. The "Achilles" is a concrete illustration of such approach and reach of the limit.

Eight years after the issue of du Bois-Reymond's book there appeared a posthumous monograph prepared by a privat-docent in philosophy at the University of Strassburg, Dr. Benno Kerry. It is a philosophic study of the theory of limits.[4] Familiar with the book of du Bois-Reymond, as well as with

[1] P. du Bois-Reymond, *op. cit.*, p. 149.
[2] P. du Bois-Reymond, *op. cit.*, pp. 122, 123.
[3] P. du Bois-Reymond, *op. cit.*, pp. 140, 141.
[4] Dr. Benno Kerry, *System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnisstheorie.* Herausgegeben v. Gustav Kohn, Leipzig u. Wien, 1890.

the work of Dedekind and G. Cantor, Kerry produced a very illuminating publication. Neither Kerry nor du Bois-Reymond discuss Zeno directly.

In 1885 a discussion of Zeno's arguments against motion was opened up in France, which lasted over a decade and in which a large number of writers participated. At no other decade and in no other country was the discussion of the topic so persistent and general. Never before have philosophical writers based their solutions of Zeno's puzzles so persistently upon the postulate of the discontinuity of time and space. Among the participants in the discussion were the mathematicians Paul Tannery, G. Milhaud, G. Frontera, G. Mouret, and L. Couturat, but the leading part was taken by the French philosophers Ch. Renouvier, F. Evellin, G. Noël, V. Brochard, and G. Lechalas.

In 1885 the noted historian of mathematics, Paul Tannery, published an article, *Le concept scientifique du continu. Zénon d'Elée et Georg Cantor*,[1] which we noted at the beginning of this history as advancing a brilliant and novel explanation of the *purpose* of Zeno's arguments. It concluded with an account of Cantor's continuum. This article was supported by a paper from the pen of Gaston Samuel Milhaud, who was successively professor of mathematics in the lycées at Nizza, Havre, Lille and Montpellier and who has been, since 1895, professor of philosophy at the University of Montpellier. Otherwise Tannery's article received little attention. The discussions carried on during the decade do not center in Cantor's continuum. Cantor's ideas were then very recent and not yet fully elaborated. Milhaud's article above referred to bore the title, *Le concept du nombre chez les Pythagoriciens et les Eléates*. Before this Milhaud had written a philosophical dialogue on *La notion de limite en mathématiques*,[2] which avowedly aimed to throw light on the "Achilles." *A* and *B* discuss the subject. *A* declares that the mind cannot encompass at once an unlimited series of elements; *B* declares that this negative quality of our mind has nothing to do with the existence or non-existence of an attainable limit. Whether the limit is actually reached or not is of no interest to the mathematician and is foreign to mathematics. Though a clever dialogue, it was criticized by Ernest Jean Georges Mouret, ingénieur-en-chef de ponts et chaussées, for dodging the real issue in the "Achilles" and failure to remove the underlying difficulties.[3]

Victor Brochard wrote on Zeno in 1885, in a memoir to l'Académie des Sciences morales et politiques and again in 1893, in a paper entitled, *Les prétendus sophismes de Zénon d'Élée*.[4] He is mainly concerned with the question of Zeno's purpose and favors traditional views rather than the views of Tannery and Milhaud. In the earlier publication Brochard expressed the belief that Zeno meant actually to deny all motion. In the later article he prefers to speak of Zeno's *arguments* rather than Zeno's *sophisms*.

[1] *Revue philosophique de la France et de l'étranger*. Dixième année, XX, 1885, Paris, pp. 385–410. Excepting the part on G. Cantor, this article is reproduced almost verbatim in P. Tannery's *Pour l'histoire de la science hellène*, Paris, 1887, Chapt. X, pp. 247–261.

[2] *Revue philosophique* [Th. Ribot], Vol. 32, Paris, 1891, p. 1.

[3] "Le problème d'Achille" by George Mouret, in *Revue philosophique*, Vol. 33, Paris, 1892, p. 67.

[4] *Revue de métaphysique et de morale*, I, 1893, Paris, pp. 209–215.

A pamphlet, entitled, *Étude sur les arguments de Zénon d'Élée contre le mouvement,* Paris, 1891, was published by G. Frontera, in the preface of which he states that he was induced to prepare this study by the fact that some young students of philosophy informed him that in certain lectures on philosophy which they had attended it was seriously taught that the "Achilles" could not be explained and that motion was an illusion of the senses. Frontera holds the views of a mathematician untouched by the ideas of G. Cantor. He speaks in the "Dichotomy" of an infinite *series* of parts, instead of an infinite *number* of parts, "parce que l'idée de nombre exclut l'idée de l'infini." He says that Zeno considered only *space,* while the subject calls for the consideration of three things, *space, time, velocity.* The sum of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., of a distance cannot be infinite as Zeno claimed, but must be finite. In dealing with series, Frontera reveals no knowledge of St. Vincent and other writers on the "Achilles," except the contemporary French writers, V. Brochard, F. Evellin. He speaks in high terms of Evellin's work, *Infini et quantité.* He writes down the convergent series for the time required to overtake the tortoise. He rightly insists that the velocities of Achilles and the tortoise must be maintained to the end, that Zeno does not do this and arrives at an absurd conclusion by the tacit assumption of chimerical conditions. The "Stade" he considers simply a question of relative motion. G. Mouret[1] criticizes Frontera for passing from the series to its limit without perceiving that it is precisely at this point where the difficulty lies. Mouret declares that the "Achilles" constitutes really a criticism of the foundations of convergent series and of the infinitesimal calculus. Frontera made reply,[2] emphasizing his former contention that the difficulty with the "Achilles" has been the chimerical hypothesis introduced by Zeno and his followers for now nearly 2,500 years.

Exception to Mouret's remark that Zeno's arguments were a criticism of the fundamental principles of the calculus and of series was taken by Louis Couturat.[3] Mouret's claim that Zeno arbitrarily excludes the point of meeting in the "Achilles" is historically not a correct interpretation; Zeno claimed that, to arrive at a point of meeting, Achilles had to run through an infinity of spaces, which Zeno thought impossible. The nerve of Zeno's argument consists, according to Couturat, in the axiom which even to-day is accepted as evident by many good heads: "The actual infinite cannot be realized"; any movement must contain an actual infinity of parts, which Zeno declared to be impossible. Zeno's sophism, if it exists, is more subtle and more profound than Mouret realizes; it is not a gross paralogism the falsity of which at once leaps into view.

In the year 1893 the discussion continued unabated. Georges Noël contributed an article, *Le mouvement et les arguments de Zénon d'Élée,*[4] which in many respects is an able discussion, but the author unfortunately does not always

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[1] *Revue philosophique,* Vol. 33, 1892, p. 67.
[2] *Revue philosophique,* Vol. 33, 1892, p. 311.
[3] *Revue philosophique,* Vol. 33, 1892, p. 314.
[4] *Revue de métaphysique et de morale,* I, 1893, pp. 107–125.

distinguish between what is assumed and what is the result of clear logic, with that sharpness that is necessary to clear up such a difficult subject as motion; the article is marred by hidden assumptions. Noël claims that as soon as one grasps the true nature of movement, the refutation of Zeno is easy. Zeno divided the displacement into an infinitely increasing number of smaller displacements, standing out as distinct events which must concur in producing the final event. These smaller displacements, infinite in number, are not all given; hence Zeno concludes that the final event cannot take place. But, says Noël, the smaller displacements are not at all veritable conditions for the occurrence of the final event. They are coördinate events, but not subordinate. The final event derives its raison d'être direct from the state of the motion and its velocity. Thereby all positions taken by the moving point are given and all are on an equal footing. Their order of succession in time is in no way an order of logical dependence. Logically they are all given with the motion. Little does it matter if they are infinite in number. Neither the moving body, nor the mind contemplating it is really bound to number them. They do not introduce any real divisions in the motion. It is one motion, and its continuity excludes all actual division. The particular motions do not exist, except from an arbitrary subjective view point. Thus the "Dichotomy" and "Achilles" are real sophisms, but of such a nature that the human mind is almost inevitably carried away by them, as long as it neglects to subject the fundamental principles to a critical analysis. Motion is not a succession of positions, it is a "becoming."

Noël attacks the advocates of discontinuity, such as Evellin, who postulate the existence of a minimum distance and minimum time. Noël argues that such minimum existence is disproved by the "Arrow" and the "Stade."

Many years before this, Ch. Renouvier[1] and F. Evellin[2] argued against the actual infinite and in favor of the discontinuity of space and time; the number of parts in which the path AB can be divided is either finite or infinite, but the number cannot be infinite without the inexhaustible being found exhausted, hence the number is finite. Besides his book on *Infini et quantité,* Evellin published articles in 1893[3] and 1894[4] in which he elaborates his ideas with reference to our topic. Noël's explanation of the nature of motion does not appeal to him; Noël's hypothesis of motion as a "becoming" affords only a moment's illusion of having escaped the difficulties set by the dialectician of Elea. Evellin declares that Zeno's four arguments present two branches of a dilemma. The "Dichotomy" and "Achilles" are aimed at the mathematicians with their infinity, the "Arrow" and "Stade" are aimed at the partisans of limited division. Making negation of the infinite, he explains Zeno by assuming the discontinuity of time and space and the existence of indivisible parts of space and

[1] *Esquisse d'une classification systematique des doctrines philosophiques*, T. I, Paris, 1885. Renouvier's proof of the non-existence of the actual infinite, as well as other proofs of this are critically examined by Georg Cantor in his article, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitschr. f. Philosophie und philosophische Kritik*, Bd. 91. Separatabdruck, Halle, p. 22.

[2] François Evellin, *Infini et quantité*, 1880.

[3] "Encore a propos de Zénon d'Élée," *Revue de métaphysique et de morale*, I, p. 382.

[4] "La divisibilité dans la grandeur," *Revue de métaphysique et de morale*, II, pp. 129–152.

time. On the hypothesis of unlimited divisibility he claims that Zeno successfully disproved motion. The concept of a line, says Evellin, must be such as to explain motion. The line must have parts; motion exhausts the line and therefore also its parts; hence, those parts have a number, but this number escapes us. Each moment of motion must mark an advance, but there is no advance except in the exhaustible and the finite. The "Dichotomy" and "Achilles" make this very plain. Evellin denies that the "Stade" disproves the possibility of indivisibles.

a	b	c			a	b	c	
a_1	b_1	c_1		a_1	b_1	c_1		
a_2	b_2	c_2				a_2	b_2	c_2

If in an indivisible moment the indivisible lines represented by a, b, c, a_1, etc., shift, so that a_2 comes to c_1, the question arises, how can a_2 and b_1 have found time to meet each other? They must indeed pass each other, if the discontinuity of space is assumed, but they need not meet at all in case of discontinuity.

Another champion of discontinuity is Georges Lechalas. In 1895 he brought out a book, an *Étude sur l'espace et le temps*, Paris, of which an enlarged edition appeared in 1910. In 1893 he contributed a journal article on Zeno.[1] He proceeds on the maxim that, so far as the realized number of things in the actual world is concerned, number always means *finite* number. He distinguishes abstract from realized number. He made a study of G. Cantor and offers no objection to actual infinity, provided this concept is confined to abstract number. He claims contradiction in its realization. All realized aggregates, he says, can be counted by the ordinary process. He denies the possibility of a real continuum and hence concludes that both space and time are discontinuous. While agreeing with Evellin on matters of discontinuity, Lechalas disagrees with him on the existence of a minimum distance and a minimum time. Evellin's mode of escape from the clutches of the "Stade" does not satisfy him. If the atom is indivisible, says Lechalas, this is not a minimum of extension; extension is not a property of atoms, but a relation between them. A point in moving from one position to another occupies only a finite number of intermediate positions; hence, in the "Stade," a_2 and b_1 may not be in the same vertical line at any time. Using discontinuity as a magic wand, he gets a_2 to c_1 without meeting b_1.

[To be concluded in the November issue.]

[1] "Note sur les arguments de Zénon d'Élée" in *Revue de métaphysique et de morale*, I, 1893, p. 396.

attention of college men fixed upon the varied and special problems of college work in mathematics, and with many group organizations for interchange of ideas, a new era will be inaugurated in this field.

3. **The Stimulation of National Organization.** The most natural climax of wide group organization is a national organization, such as was referred to in the October issue, and conversely, such a national organization will react on the formation of smaller groups and both will provide a far-reaching stimulus to individual activity. The MONTHLY would have welcomed the incorporation of such an organization within the American Mathematical Society, but since this is not to be, we look forward with high hopes and great enthusiasm to the organization of a new national society, and with more than four hundred charter members, we see no reason why commendable things should not be accomplished through this movement for the cause of mathematics in America.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI Colorado College.

X.

E. POST-CANTORION DISSENSIONS (Concluded).

With the advent of the new century, discussion on Zeno began to quiet down in France. We note only two articles. In 1907 O. Hamelin wrote on the "Arrow," but the interest of his article centers in what constitutes the most probable renderings of the Aristotelian text.[1] In 1909 a novel attempt to solve Zeno's puzzles was made by Dunan[2] in an article in which he retracts what he said on this subject in a pamphlet of 1884.[3]

He believes that the difficulties vanish, on the recognition that motion takes place through a space, one and indivisible, without succession and parts. He admits that such a proposition raises considerable difficulty, which cannot be removed except by long and elaborate metaphysics, of which he gives in his article only a bare sketch.

No less radical is the position of Henri Bergson. He holds that philosophy must get back to reality itself. Reality is supplied by intuition. Pure intuition, external or internal, is that of undivided continuity. Every movement, in as much as it is a passage from rest to rest, is in fact absolutely indivisible. Sight perceives the movement in the form of a line which is traversed, and this line, like all space, may be indefinitely divided. We must not confound the data of

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[1] *L'année philosophique* de F. Pillon, Paris, 1907, pp. 39–44.

[2] "Zénon d'Élée et le Nativism" in *Annales de Philosophie Chrétienne*, 1909.

[3] *Les arguments de Zénon d'Élée contre le mouvement*, Nantes, 1884.

the senses, which perceive the movement as an undivided whole, with the artifice of the mind which divides into parts the path traversed. Says Bergson:[1]

"You substitute the path for the journey, and because the journey is subtended by the path you think that the two coincide. But how should a *progress* coincide with a *thing*, a movement with an immobility? . . . And from the fact that this line is divisible into parts and that it ends in points, we cannot conclude either that the corresponding duration is composed of separate parts or that it is limited by instants. The arguments of Zeno of Elea have no other origin than this illusion. They all consist in making time and movement coincide with the line which underlies them, in attributing to them the same subdivisions as to the line, in short in treating them like that line. In this confusion Zeno was encouraged by common sense, which usually carries over to the movement the properties of its trajectory, and also by language, which always translates movement and duration in terms of space. . . . But the philosopher who reasons upon the inner nature of movement is bound to restore to it the mobility which is its essence, and this is what Zeno omits to do. By the first argument (the Dichotomy) he supposes the moving body to be at rest, and then considers nothing but the stages, infinite in number, that are along the line to be traversed: we cannot imagine, he says, how the body could ever get through the interval between them. But in this way he merely proves that it is impossible to construct, *à priori*, movement with immobilities, a thing no man ever doubted. The sole question is whether, movement being posited as a fact, there is a sort of retrospective absurdity in assuming that an infinite number of points has been passed through. But at this we need not wonder, since movement is an undivided fact, or a series of undivided facts, whereas the trajectory is infinitely divisible. In the second argument (the Achilles) movement is indeed given, it is even attributed to two moving bodies, but, always by the same error, there is an assumption that their movement coincides with their path, and that we may divide it, like the path itself, in any way we please. Then, instead of recognizing that the tortoise has the pace of a tortoise and Achilles the pace of Achilles, so that after a certain number of these indivisible acts or bounds Achilles will have outrun the tortoise, the contention is that we may disarticulate as we will the movement of Achilles and, as we will also, the movement of the tortoise: thus reconstructing both in an arbitrary way, according to a law of our own which may be incompatible with the real conditions of mobility. The same fallacy appears, yet more evident, in the third argument (the Arrow) which consists in the conclusion that, because it is possible to distinguish points on the path of a moving body, we have the right to distinguish indivisible moments in the duration of its movement. But the most instructive of Zeno's arguments is perhaps the fourth (the Stadium) which has, we believe, been unjustly disdained, and of which the absurdity is more manifest only because the postulate masked in the three others is here frankly displayed. Without entering on a discussion which would here be out of place, we will content ourselves with observing that motion, as given to spontaneous perception, is a fact which is quite clear, and that the difficulties and contradictions pointed out by the Eleatic school concern far less the living movement itself than a dead and artificial reorganization of movement by the mind."

Bergson discusses the "Arrow" more fully in his *L'Evolution creatrive*, 1907, where he refers[2] to the absurdity of regarding movement as made up of immobilities. He says:

"Philosophy perceived this as soon as it opened its eyes. The arguments of Zeno of Elea although formulated with a different intention, have no other meaning. . . . Motionless in each point of its course, it is motionless during all the time of its moving. Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course. The most that we can say is that it might be there, in this sense, that it passes there and might stop there. . . . You fix a point C in the interval passed, and say that at a certain moment the arrow was at C. If it had been there it would have been stopped there, and you would no longer have had a flight from A to B, but *two* flights, one from A to C and the other from C to B, with an interval of rest. A single movement is entirely, by the hypothesis, a movement between two stops; if there are intermediate stops, it is no longer a single movement."

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[1] H. Bergson, *Matter and Memory*, transl. by Nancy M. Paul and W. Scott Palmer, London, 1911, pp. 248, 250–253. The first French edition appeared in 1896.

[2] H. Bergson, *Creative Evolution*, transl. by A. Mitchell, London, 1911, pp. 325–327.

There have been many dissussions of Bergson. One writer endeavors to point out his errors by returning to the continuums of Artistotle and Thomas Aquinas.[1] Most pertinent to our topic are the criticisms by Bertrand Russell, of Cambridge, England, which are displayed by the following quotations:[2]

". . . it will be said, the arrow is where it is at any one moment, but at another moment it is somewhere else, and this is just what constitutes motion. Certain difficulties, it is true, arise out of the continuity of motion, if we insist upon assuming that motion is also discontinuous. These difficulties, thus obtained, have long been part of the stock-in-trade of philosophers. But if, with the mathematicians, we avoid the assumption that motion is also discontinuous, we shall not fall into the philosopher's difficulties. A cinematograph in which there are an infinite number of films, and in which there is never a *next* film because an infinite number come between any two, will perfectly represent a continuous motion. Wherein, then, lies the force of Zeno's argument? . . . Zeno assumes, tacitly, the essence of the Bergsonian theory of change. That is to say, he assumes that when a thing is in process of continuous change, even if it is only change of position, there must be in the thing some internal *state* of change. The thing must, at each instant, be intrinsically different from what it would be if it were not changing. He then points out that at each instant the arrow simply is where it is, just as it would be if it were at rest. Hence he concludes that there can be no such thing as a *state* of motion, and therefore, adhering to the view that a state of motion is essential to motion, he infers that there can be no motion and that the arrow is always at rest. Zeno's argument, therefore, though it does not touch the mathematical account of change, does, *prima facie*, refute a view of change which is not unlike M. Bergson's. How, then, does M. Bergson meet Zeno's argument? He meets it by denying that the arrow is ever anywhere. After stating Zeno's argument, he replies: 'Yes, if we suppose that the arrow can ever *be* in a point of its course. Yes again, if the arrow, which is moving, ever coincides with a position, which is motionless. But the arrow never *is* in any point of its course.' (C. E., p. 325.) This reply to Zeno, or a closely similar one concerning Achilles and the Tortoise, occurs in all his three books. Bergson's view plainly, is paradoxical; whether it be *possible*, is a question which demands a discussion of his view of duration. His only argument in its favor is the statement that the mathematical view of change 'implies the absurd proposition that movement is made of immobilities.' (C. E., p. 325.) But the apparent absurdity of this view is merely due to the verbal form in which he has stated it, and vanishes as soon as we realize that motion implies relations. A friendship, for example, is made out of people who are friends, but not out of friendships. . . . So a motion is made out of what is moving, but not out of motions. It expresses the fact that a thing may be in different places at different times, and that the places may still be different however near together the times may be. Bergson's argument against the mathematical view of motion, therefore, reduces itself, in the last analysis, to a mere play upon words." . . .

"Mathematics conceives change, even continuous change, as constituted by a series of states; Bergson, on the contrary, contends that no series of states can represent what is continuous, and that in change a thing is never in any state at all." . . .

"One of the bad effects of an anti-intellectual philosophy, such as that of Bergson, is that it thrives upon the errors and confusions of the intellect. Hence it is led to prefer bad thinking to good, to declare every momentary difficulty insoluble, and to regard every foolish mistake as revealing the bankruptcy of intellect and the triumph of intuition. . . . As regards mathematics, he has deliberately preferred traditional errors in interpretation to the more modern views which have prevailed among mathematicians for the last half century."

Thus it is seen that among recent French philosophers the Cantor continuum has been neglected and no satisfactory substitute has been advanced.

A treatment of the "Achilles" altogether different from that hitherto given by either philosophers or mathematicians is given by Russell.[3] After explaining infinite number and the modern continuum, he says in the *International Monthly:*

[1] T. J. Gerrard, *Bergson, an Exposition and Criticism*, London and Edinburgh, 1913, pp. 23 ff.

[2] B. Russell, "The Philosophy of Bergson," *The Monist*, July, 1912. Russell's references (C. E.) are to the English translations of Bergson's *Creative Evolution*.

[3] See B. Russell, *Principles of Mathematics*, 1902; "Recent work on the Principles of Mathematics" in the *International Monthly*, Vol. IV, 1901, pp. 83–101.

"We can now understand why Zeno believed that Achilles cannot overtake the tortoise and why as a matter of fact he can overtake it. We shall see that all the people who disagreed with Zeno had no right to do so, because they all accepted premises from which his conclusion followed. . . . Then he [Achilles] will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another place in another moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise should have been, would be only part of the places, where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion. The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities."

The conjectures which Russell makes on the history of the "Achilles" are, in the main, without foundation. There is no historical evidence for believing that Zeno based the "Achilles" on the doctrine that the whole is greater than any of its parts. Aristotle bases Zeno's argument on the assertion that a line or distance cannot be reduced by any process of successive division to elements that are mathematical points. Russell's version of the paradox is what Zeno might have said, but did not actually say. It is far simpler to explain than is that of Zeno. Assent can be readily secured to the fact that, in infinite aggregates, the whole is not greater than certain of its parts.[1]

That Russell's argument, though correct in itself, does not meet the exact difficulty experienced by many persons, is brought out by C. D. Broad,[2] who points out that the "Achilles" rests on the false assumption that "what is beyond every one of an infinite series of points, must be infinitely beyond the first point of the series." Broad considers it important even at this time to settle this controversy, "because it and Zeno's other paradoxes have become the happy hunting-ground of Bergsonians and like contemners of the human intellect." What makes infinite divisibility a stumbling block to so many is the fact that appeal is made to sensory intuition and imagination—the very faculty of the mind which proves itself unable to cope with the problem. But our powers of analysis penetrate realms of thought beyond the reach of the imagination, and it is in that territory that the arguments of Zeno are made to surrender their mysteries.

B. Russell took great interest also in the "Arrow." In the *International Review* he remarked:

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[1] In this connection a story told by De Morgan may be of interest. He relates "a tradition of a Cambridge professor who was once asked in a mathematical discussion, 'I suppose you will admit that the whole is greater than its part?' and who answered, "not I, until I see what use you are going to make of it.' " The danger of unintended implications is illustrated by an author who remarked that Gibbon always had a copy of Horace in his pocket and often in his hand, from which it would seem to follow that Gibbon's hand was sometimes in his pocket.

[2] *Mind*, Vol. 38, 1913, pp. 318, 319.

"Weierstrass, by strictly banishing from mathematics the use of infinitesimals, has at last shown that we live in an unchanging world, and that the arrow in its flight is truly at rest. Zeno's only error lay in inferring (if he did infer) that, because there is no change, therefore the world is in the same state at any one time as at any other. . . . Weierstrass has been able, by embodying his views in mathematics, where familiarity with truth eliminates the vulgar prejudices of common sense, to invest Zeno's paradoxes with the respectable air of platitudes." Elsewhere Russell expresses much the same idea by the statement that "a variable does not vary."[1]

That a variable cannot reach its limit is still widely held. In 1907 R. B. Haldane presented this as the teaching of mathematics, in a presidential address to the Aristotelian Society, entitled "The Methods of Modern Logic and the Conception of Infinity." In a review of this address, B. Russell says[2] that this property "belongs to limits of a certain particular sort," which constitute "an extremely special case, not realized in most of the series in which limits exist."

The creation of the theory of sets and of the Cantor continuum lead to modified definitions of the limit. In this theory the concept of a limit was divorced from the idea of quantity and measurement. The question whether the variable reaches its limit or not is ignored as being of no interest. Whether it reaches its limit or not depends upon the nature of the variation in a particular case; the sequence of values may include the limit, or it may not. The *limiting point* of a set of points is one for which every interval, however small, containing the limiting point, encloses a point of the set, distinct from the limiting point itself. A *limit* is merely the arithmetical equivalent of the limiting point in geometry. The introduction of a transfinite number as a limit has carried with it still further modification of the idea of a limit. Small intervals do not fit here. Says Bertrand Russell: "If we consider the whole series of integers, finite and infinite, arranged in order of magnitude, then the class of finite integers, considered as part of this series, has an upper limit, namely the smallest of the infinite integers (which is the number of finite integers)." Here there is no "negligible difference" between variable and limit; "the difference between the finite integers and their limit remains constant and infinite." Again he says: "A limit must not be conceived as something to which the successive terms of the class approach indefinitely near; they may all be at an infinite distance from the limit, or at a distance which remains permanently greater than some given finite distance; or the series concerned may be one in which there is no such thing as distance or difference." His definition of a limit is as follows: "Given any series, and a class α of terms belonging to the series, a term x belonging to the series is called the *upper limit* of α if every term of α precedes x, and every term of the series which precedes x precedes some member of α." He gives a similar definition for *lower limit*.[3] It is to be observed that the modern definitions of a limit are free from the concept of the old-time infinitesimal.

As now we pause and look backward, we see that a full and logically correct explanation of Zeno's arguments on motion has been given by the philosophers of mathematics. Looking about us, we see that the question is still regarded as

[1] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, p. 240.
[2] *Mind*, Vol. 33, 1908, p. 239.
[3] B. Russell, "Mr. Haldane on Infinity," *Mind*, Vol. 33, London, 1908, pp. 240, 241.

being in an unsettled condition. Philosophers whose intellectual interests are remote from mathematics are taking little interest in the linear continuum as created by the school of Georg Cantor. Nor do they offer a satisfactory substitute. The main difficulty is not primarily one of logic; it is one of postulates or assumptions. What assumptions are reasonable and useful? On this point there is disagreement. Cantor and his followers are willing to assume a continuum which transcends sensuous intuition. Others are not willing to do so. Hence the divergence. In the Koran there is a story that, after the creation of Adam, the angels were commanded to make him due reverence. But the chief of the angels refused, saying: "Far be it from me a pure spirit to worship a creature of clay." For this refusal he was shut out from Paradise. The doom of that chief, so far as the mathematical paradise is concerned, awaits those who refuse to examine with proper care the massive creation by our great mathematicians, without which the tiniest quiver of a leaf on a tree remains incomprehensible.[1]

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MATHEMATICAL MEETINGS IN CALIFORNIA.

I. THE TWENTY-SECOND SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The American Mathematical Society met for its twenty-second summer meeting as announced by the Society, on August 3, 1915, at the University of California in Berkeley. The first meeting was in conjunction with Section A of the American Association for the Advancement of Science, on Tuesday morning. Professor Keyser, of Columbia University, delivered an address on the human significance of mathematics, and Director Hale, of the Mount Wilson Solar Observatory, delivered an address on the work of a modern observatory. The attendance was very large and included members of the American Mathematical

[1] Since the completion of this article there has appeared Bertrand Russell's *Our Knowledge of the External World as a Field for Scientific Method in Philosophy*, Open Court Company, 1914, in which much attention is given to Zeno's arguments. An article on Zeno by Philip E. B. Jourdain will soon appear in *Mind*.